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RAMIFIED DEGENERATE PRINCIPAL SERIES REPRESENTATIONS FOR $Sp(n)$

BY

STEPHEN S. KUDLA*

University of Maryland
College Park, MD 20742

AND

STEPHEN RALLIS**

Ohio State University
Columbus, Ohio 43210

ABSTRACT

In this paper we give a complete description of the points of reducibility, components and composition series of the degenerate principal series representations of the group $Sp(n, F)$, F a non-archimedean local field, which are induced from a character of a maximal parabolic subgroup $P = MN$ with Levi subgroup $M \simeq GL(n, F)$. We show that all of the reducibility is accounted for by submodules coming from the Weil representation associated to quadratic forms over F. The local results of this paper play an essential role in our extension of the Siegel–Weil formula relating theta integrals and special values of Eisenstein series.

Introduction

In this paper we will give a complete description of the points of reduction and the constituents of a certain family of induced representations of the symplectic group $G = \mathrm{Sp}_n(F)$ over a non-archimedean local field F of characteristic zero.

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More precisely, recall that G has a maximal parabolic subgroup of the form $P = MN$ with Levi factor $M \simeq \mathrm{GL}_n(F)$ and unipotent radical $N \simeq \mathrm{Sym}_n(F)$. For any unitary character χ of $F^\times$ and for any $s \in \mathbb{C}$, we consider the representation

$$I(s,\chi) = \mathrm{Ind}_P^G \chi \cdot |\ |^s$$

induced from the character $m \mapsto \chi(\det m)|\det m|^s$, where the induction is normalized so that $I(s,\chi)$ is naturally unitarizable when s is pure imaginary. Such representations play a central role in our work on the Weil-Siegel formula [**9**, **10**,**12**,**13**] and hence, ultimately, in the study of the special values of certain Langlands L-functions [**2**,**5**]. In the real case, fairly complete information about the points of reducibility and about certain constituents of the $I(s,\chi)$'s was obtained in [**11**], although the precise composition series was not determined. In the non-archimedean case, the points of reducibility and a complete description of the constituents and composition series was given by Gustafson [**3**] provided the character χ is *unramified*. Unfortunately, in global applications ramified characters will arise, and the method of [**3**] cannot be applied.

In this paper we determine all of the points of reducibility for an arbitrary character χ. More precisely, we have

THEOREM: *Assume that χ is normalized as explained in section 1.*

(i) *If $\chi^2 \neq 1$, then $I_n(s,\chi)$ is irreducible for all s.*

(ii) *If $\chi^2 = 1$ but $\chi \neq 1$, then $I_n(s,\chi)$ is irreducible whenever s does not lie in the set*

$$\{\ \pm(-\frac{n+1}{2}+k)+\frac{i\pi}{\log q}\mathbb{Z} \mid 1 \le k \le \left[\frac{n}{2}\right]\} \cup \begin{cases} \frac{i\pi}{\log q}\mathbb{Z} & \text{when } n \text{ is odd} \\ \phi & \text{when } n \text{ is even.} \end{cases}$$

(iii) *If $\chi = 1$, then $I_n(s,\chi)$ is irreducible whenever s does not lie in the set*

$$\{\pm(-\frac{n+1}{2}+k)+\frac{i\pi}{\log q}\mathbb{Z} \mid 1 \le k \le \left[\frac{n}{2}\right]\} \cup \{\pm\frac{n+1}{2}+\frac{2\pi i}{\log q}\mathbb{Z}\} \cup \begin{cases} \frac{i\pi}{\log q}\mathbb{Z} & \text{when } n \text{ is odd} \\ \phi & \text{when } n \text{ is even.} \end{cases}$$

Next we describe constituents of the $I_n(s,\chi)$'s which are attached to quadratic forms and which account for all of the reducibility at points allowed in the previous result. If V, $(\ ,\ )$ is a non-degenerate inner product space of dimension m over F, and if m is even, then there is a subrepresentation $R_n(V) \subset I_n(s_0,\chi_V)$

associated to V, where $s_0 = \frac{m}{2} - \frac{n+1}{2}$ and $\chi_V(x) = (x, (-1)^{m/2}\det(V))_F$. Here $(\ ,\ )_F$ is the Hilbert symbol for the field F and $\det(V) = \det((v_i, v_j))$ for any basis $\{v_i\}$ of V over F. The representation $R_n(V)$ (which need not be irreducible) may be viewed as the image of the trivial representation of $O(V)$, the orthogonal group of V, under the local theta correspondence. For convenience we state the result in the cases $\chi \neq 1$ and $\chi = 1$ separately.

THEOREM: *Assume that $\chi^2 = 1$ but that $\chi \neq 1$. Let $s_0 = \frac{m}{2} - \frac{n+1}{2}$, with $2 \leq m \leq 2n$ and m even. Let V_1 and V_2 be the two inequivalent quadratic spaces over F with $\dim V_i = m$, $\chi_{V_i} = \chi$. They are distinguished by their Hasse invariants.*

(i) *If $2 \leq m < n+1$ so that $s_0 < 0$, then $R_n(V_1)$ and $R_n(V_2)$ are irreducible, $R_n(V_1) \oplus R_n(V_2)$ is a submodule of $I_n(s_0, \chi)$ and the quotient*

$$I_n(s_0,\chi)/(R_n(V_1) \oplus R_n(V_2))$$

is irreducible.

(ii) *If $m = n+1$ (hence n is odd), so that $s_0 = 0$, then $R_n(V_1)$ and $R_n(V_2)$ are irreducible and*

$$I_n(s_0,\chi) = R_n(V_1) \oplus R_n(V_2).$$

(iii) *If $n+1 < m \leq 2n$, then $R_n(V_1)$ and $R_n(V_2)$ are maximal submodules of $I_n(s_0,\chi)$ and $R_n(V_1) \cap R_n(V_2)$ is irreducible.*

THEOREM: *Assume that $\chi = 1$, and let $s_0 = \frac{m}{2} - \frac{n+1}{2}$, with $0 \leq m \leq 2n+2$ and m even. Let V_1 be the split quadratic space of dimension m. Also, if $4 \leq m \leq 2n+2$, let V_2 be the quaternionic quadratic space of dimension m (see section 3 for the terminology).*

(i) *If $m = 0$ or 2, then $R_n(V_1)$ is irreducible and is the maximal submodule of $I_n(s_0, 1)$. In the case $m = 0$, $R_n(V_1)$ is the space of constant functions.*

(ii) *If $4 \leq m < n+1$ so that $s_0 < 0$, then $R_n(V_1)$ and $R_n(V_2)$ are irreducible, $R_n(V_1) \oplus R_n(V_2)$ is a submodule of $I_n(s_0, 1)$ and the quotient*

$$I_n(s_0,1)/(R_n(V_1) \oplus R_n(V_2))$$

is irreducible.

(iii) *If $m = n+1$ (hence n is odd), so that $s_0 = 0$, then $R_n(V_1)$ and $R_n(V_2)$ are irreducible and*

$$I_n(s_0,1) = R_n(V_1) \oplus R_n(V_2).$$

(iv) *If* $n+1 < m \leq 2n-2$, *then* $R_n(V_1)$ *and* $R_n(V_2)$ *are maximal submodules of* $I_n(s_0, 1)$ *and* $R_n(V_1) \cap R_n(V_2)$ *is irreducible.*

(v) *If* $m = 2n$ *or* $2n+2$, *then* $R_n(V_1) = I_n(s_0, 1)$, $R_n(V_2)$ *is a maximal submodule of* $I_n(s_0, 1)$ *and* $R_n(V_2)$ *is irreducible.*

In fact, the various subquotients may all be identified. For example, when $m = 2n+2$ and $\chi = 1$, then the quotient $I_n(\frac{n+1}{2}, 1)/R_n(V_2) = R_n(V_1)/R_n(V_2)$ is isomorphic to the trivial representation on the constant functions in $I_n(-\frac{n+1}{2}, 1)$. Similarly, for $m = 2n$, the quotient $I_n(s_0, 1)/R_n(V_2) = R_n(V_1)/R_n(V_2)$ is isomorphic to the irreducible submodule $R_n(V_{1,0})$ of $I_n(-s_0, 1)$ associated to the split binary form $V_{1,0}$. In general when $\chi = 1$ and $n+1 < m \leq 2n-2$, let V_1 and V_2 be as in the previous theorem, and let $V_{1,0}$ and $V_{2,0}$ be the split and quaternionic quadratic spaces of dimension $2n+2-m$ respectively. Then

$$R_n(V_1)/(R_n(V_1) \cap R_n(V_2)) \simeq R_n(V_{1,0})$$

and

$$R_n(V_2)/(R_n(V_1) \cap R_n(V_2)) \simeq R_n(V_{2,0}).$$

Analogous results hold when $\chi \neq 1$.

We also describe the image and kernel of the normalized intertwining operator

$$M_n^*(s, \chi) : I_n(s, \chi) \longrightarrow I_n(-s, \chi^{-1})$$

in all cases (sections 5 and 6), and we determine all of the exponents, with respect to the Borel subgroup of $\mathrm{Sp}_n(F)$, of the representations $I_n(s, \chi)$, $R_n(V)$ and their various constituents (sections 4 and 6). For example, the representation $R_n(V_1) \cap R_n(V_2)$, which occurs as an irreducible submodule to the right of the unitary axis, turns out to have multiplicity free exponents. This rather striking fact depends on a somewhat tricky combinatorial fact about shuffles (Proposition 6.2).

Finally, it follows from our results that the representation $R_n(V)$ always has a unique irreducible quotient. This is the Howe duality conjecture [**6**], [**21**] in our rather special situation, but without any restriction on the residue characteristic.

The results of this paper are a necessary technical background for our forthcoming work on a general Weil-Siegel formula in the divergent range. They may also be seen as a kind of analogue of this formula over a local field.

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1. Preliminaries

In this section we will set up the basic machinery which will be used throughout the paper. Our notation will be mostly that of [9,10,13].

Let F be a non-archimedean local field, which, for convenience, we assume to have characteristic 0. Let $\mathcal{O}$ be the ring of integers of F, and let $\mathcal{P} = \varpi\mathcal{O}$ be its maximal ideal with a fixed generator ϖ. Let $q = |\mathcal{O}/\mathcal{P}| = |\varpi|^{-1}$. We fix an additive character ψ of F whose conductor is $\mathcal{O}$.

Let $G = G_n = \mathrm{Sp}_n(F)$ be the symplectic group of rank n over F. Here and elsewhere we will often drop the subscript n unless it is required for some inductive argument. Let $P \subset G$ be the maximal parabolic subgroup with Levi decomposition

$$P = MN$$

where

$$M = \{m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \mid a \in \mathrm{GL}_n(F)\}$$

and

$$N = \{n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \mid b = {}^t b \in \mathrm{Sym}_n(F)\}.$$

Here $\mathrm{Sym}_n(F)$ is the space of $n \times n$ symmetric matrices. We will sometimes refer to P as the Siegel parabolic. Let $B \subset P$ be the Borel subgroup with unipotent radical

$$U = \{\ m(u)n(b) \mid u \text{ upper triangular unipotent and } b \in \mathrm{Sym}_n(F)\ \} \tag{1.1}$$

and Levi factor

$$A = \{\ m(a) \mid a = \mathrm{diag}(a_1, \ldots, a_n)\ \}. \tag{1.2}$$

We fix the maximal compact subgroup $K = \mathrm{Sp}_n(\mathcal{O})$ of G, and for the Iwasawa decomposition $G = PK = NMK$ we write $g = nm(a)k$ for $a = a(g) \in \mathrm{GL}_n(F)$. While $a(g)$ is not uniquely determined by this decomposition, the quantity $|a(g)| = |\det a(g)|$ is well defined.

Let χ be a character of $F^\times$. For $s \in \mathbb{C}$, we let $I(s,\chi) = I_n(s,\chi)$ denote the normalized smooth induced representation, consisting of smooth functions $\Phi(s)$ on G such that

$$\Phi(nm(a)g, s) = \chi(a)|a|^{s+\rho_n}\Phi(g,s). \tag{1.3}$$

Here $\rho_n = \rho_P = \frac{n+1}{2}$. A section $\Phi(s) \in I(s,\chi)$ will be called standard if its restriction to K is independent of s. Note that this restriction determines $\Phi(s)$.

As in [10,§4.], recall that we have an intertwining operator

$$M(s,\chi) : I(s,\chi) \to I(-s,\chi^{-1}) \tag{1.4}$$

defined, for $Re(s) > \rho_n$, by the integral

$$M(s,\chi)\Phi(g) = \int_N \Phi(wng,s)\,dn \tag{1.5}$$

where $w = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix} \in G$. This operator has a meromorphic analytic continuation to the whole s plane.

Occasionally in this paper we will normalize the character χ as follows. The choice of prime element ϖ yields an isomorphism

$$F^\times \simeq \mathcal{O}^\times \times \mathbb{Z}$$

and a corresponding isomorphism of character groups

$$\hat{F}^\times \simeq \hat{\mathcal{O}}^\times \times \mathbb{C}^1$$

where $\mathbb{C}^1$ is the subgroup of $\mathbb{C}^\times$ of elements of absolute value 1. We assume that under this isomorphism, χ corresponds to an element of $\hat{\mathcal{O}}^\times \times 1$, so that χ is trivial on ϖ. Note that, in particular, χ is either trivial or ramified. Of course, an arbitrary quasicharacter of $F^\times$ has the form $a \mapsto \chi(a)|a|^s$ for s in $\mathbb{C}$, so our normalization of χ does not restrict the generality of our results. The purpose of this normalization is to prevent an arbitrary translation, in the position of the poles, which could arising from a twist of χ by some power of $|\ |$.

2. A criterion for irreducibility

In this section we will compute certain Jacquet modules of $I_n(s,\chi)$ and show that this representation is irreducible for s and χ outside of a certain set.

First we have [**18**]

LEMMA 2.1: *Let N be the unipotent radical of P. Then, the Jacquet module $I_n(s,\chi)_N$ has an M stable filtration*

$$I_n(s,\chi)_N = I^0 \supset I^1 \supset \cdots \supset I^n \supset I^{n+1} = 0$$

with successive quotients

$$Z_r(s,\chi) = I^r/I^{r+1} \simeq \operatorname{Ind}_{Q_r}^{\mathrm{GL}_n(F)}(\xi_r),$$

where $Q_r \subset \mathrm{GL}(n)$ is the maximal parabolic subgroup of the form

$$\{ \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \mid a \in \mathrm{GL}(n-r),\ b \in \mathrm{GL}(r) \},$$

and ξ_r is the character of Q_r whose value on an element of the above form is

$$\chi(\det a)\chi(\det b)^{-1} |\det a|^{s+\frac{n-r+1}{2}} |\det b|^{-s+\frac{r+1}{2}}.$$

Here normalized induction is used.

Using this we obtain

PROPOSITION 2.2: *Assume that χ is normalized as explained in §1.*

(i)
$$\dim \operatorname{Hom}_G(I_n(s,\chi), I_n(-s,\chi^{-1})) \le \begin{cases} 2 & \text{if } \chi^2 = 1 \text{ and } s \in \frac{r}{2} + \frac{\pi i}{\log(q)}\mathbb{Z} \text{ for some } r \text{ with } 0 \le r < n \\ 1 & \text{otherwise.} \end{cases}$$

(ii)
$$\dim \operatorname{Hom}_G(I_n(s,\chi), I_n(s,\chi)) \le \begin{cases} 2 & \text{if } \chi^2 = 1 \text{ and } s \in \frac{\pi i}{\log(q)}\mathbb{Z} \\ 1 & \text{otherwise.} \end{cases}$$

Of course, in the second case here the dimension is equal to 1.

Proof: We simply note that

$$\operatorname{Hom}_G(I_n(s,\chi), I_n(-s,\chi^{-1})) = \operatorname{Hom}_{\mathrm{GL}_n}(I_n(s,\chi)_N, \chi^{-1}|\ |^{-s+\rho_n}),$$

and so, by Lemma 2.1,

$$\dim \mathrm{Hom}_G(I_n(s,\chi), I_n(-s,\chi^{-1})) \le \sum_{r=0}^{n} \dim \mathrm{Hom}_{\mathrm{GL}_n}(Z_r(s,\chi), \chi^{-1}|\ |^{-s+\rho_n}).$$

Next

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_n}(Z_r(s,\chi), \chi^{-1}|\ |^{-s+\rho_n}) &\simeq \mathrm{Hom}_{\mathrm{GL}_n}(\chi|\ |^{s-\rho_n}, \widetilde{Z_r(s,\chi)})\\ &\simeq \mathrm{Hom}_{\mathrm{GL}_{n-r}\times\mathrm{GL}_r}(\chi|\ |^{s-\rho_n}, \tilde{\xi}_r\cdot\delta_{Q_r}^{1/2}),\end{aligned}$$

where $\tilde{}$ denotes the contragredient representation. But

$$\tilde{\xi}_r\cdot\delta_{Q_r}^{1/2}(a,b) = \chi(\det a)^{-1}|\det a|^{-s-\rho_n+r}\chi(\det b)|\det b|^{s-\rho_n},$$

and so this last Hom is non-zero only in case either $r=n$ or $0\le r<n$ and $\chi^2(\det a)|\det a|^{2s-r}=1$. Note that our normalization of χ implies that either $\chi^2\equiv 1$ or χ^2 is non-trivial on some unit, so that no solution of the last condition exists in that case. This proves (i).

Part (ii) is proved in the same way. ∎

We use Proposition 2.2 to determine the number of possible irreducible submodules of $I_n(s,\chi)$.

PROPOSITION 2.3: *Assume that χ is normalized as explained in §1. Suppose that $\pi\subset I_n(s,\chi)$ is a G-submodule.*

(i) *If $\chi^2=1$ and*

$$s\in -\frac{n-r}{2}+\frac{i\pi}{r\log q}\mathbb{Z}$$

for some r with $1\le r\le n$, then

$$\dim \mathrm{Hom}_G(\pi, I_n(s,\chi))\le 2.$$

(ii) *Otherwise*

$$\dim \mathrm{Hom}_G(\pi, I_n(s,\chi)) = 1.$$

Proof: Given π,

$$\mathrm{Hom}_G(\pi, I_n(s,\chi)) = \mathrm{Hom}_{\mathrm{GL}_n}(\pi_N, \chi|\ |^{s+\rho_n}).$$

Now we consider the generalized eigenspaces of π_N and of $I_n(s,\chi)_N$ with respect to the action of the center of GL_n, where the eigencharacter of interest to us is

$\mu = (\chi|\ |^{s+\rho_n})^n$. First note that the central characters of the successive quotients $Z_r(s,\chi)$ of $I_n(s,\chi)_N$ are

$$z \mapsto \chi(z)^{n-2r}|z|^{(n-2r)s+r(r-n)+\frac{n(n+1)}{2}}.$$

and one of these coincides with μ if and only if

$$\chi^{2r} = 1 \qquad \text{and} \qquad -r(2s+n-r) \in \frac{2\pi i}{\log q}\mathbb{Z}.$$

Clearly $r = 0$ yields one solution, while, if $0 < r \leq n$, the condition on s is

$$s \in -\frac{n-r}{2} + \frac{i\pi}{r\log q}\mathbb{Z}.$$

For a given s this can hold for at most one r. Recalling that the filtration on $I_n(s,\chi)_N$ is decreasing, we obtain an exact sequence of generalized eigenspaces:

$$0 \longrightarrow Z_r(s,\chi)(\mu) \longrightarrow I_n(s,\chi)_N(\mu) \longrightarrow \mathbb{C}_\mu \longrightarrow 0,$$

and thus a sequence

$$0 \longrightarrow Z_r(s,\chi)(\mu) \cap \pi_N(\mu) \longrightarrow \pi_N(\mu) \longrightarrow \mathbb{C}_\mu.$$

This give us a bound

$$\dim \mathrm{Hom}_{\mathrm{GL}_n}(\pi_N, \mu) \leq 1 + \dim \mathrm{Hom}_{\mathrm{GL}_n}(Z_r(s,\chi), \mu).$$

The second term only occurs if, for our fixed s, there is an r which satisfies the above conditions. But, restricting to $\mathrm{GL}_n(\mathcal{O})$, we have

$$\begin{aligned}\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}(Z_r(s,\chi),\mu) &\leq \dim \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O})}(Z_r(s,\chi),\mu)\\ &= \dim \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O})}(\mu, Z_r(s,\chi)),\end{aligned}$$

since $\mathrm{GL}_n(\mathcal{O})$ is compact and $Z_r(s,\chi)$ is admissible.

By Frobenius reciprocity, this space is non-zero only when

$$\xi_r(a,b) = \chi(\det a)\chi(\det b)^{-1} = \chi(\det a)\chi(\det b)$$

for $a \in \mathrm{GL}_r(\mathcal{O})$ and $b \in \mathrm{GL}_{n-r}(\mathcal{O})$, i.e., when $\chi^2 = 1$. Note that we are using the fact that χ is normalized here. ∎

COROLLARY 2.4: *If the condition of part (i) of Proposition 2.3 holds, then $I_n(s,\chi)$ has at most two irreducible submodules. Otherwise, $I_n(s,\chi)$ has at most one irreducible submodule.*

We will now use Proposition 2.2 to show that $I_n(s,\chi)$ is irreducible outside of a certain set of values of s and χ. Several preliminaries will be needed.

First we recall the normalization of the intertwining operator, which is defined as in [**17**]. Let

$$a_n(s,\chi) = L(s+\rho_n - n,\chi)\prod_{k=1}^{[\frac{n}{2}]} L(2s-n+2k,\chi^2),$$

and

$$b_n(s,\chi) = L(s+\rho_n,\chi)\prod_{k=1}^{[\frac{n}{2}]} L(2s+n+1-2k,\chi^2),$$

where $L(s,\chi) = 1$ if χ is ramified, and $L(s,\chi) = (1-\chi(\varpi)q^{-s})^{-1}$ with ϖ a uniformizing parameter for F and q the order of the residue class field, when χ is unramified. Let

$$M_n^*(s,\chi) = \frac{1}{a_n(s,\chi)}M_n(s,\chi) : I_n(s,\chi) \longrightarrow I_n(-s,\chi^{-1}),$$

where $M_n(s,\chi)$ is the intertwining operator of (1.5) above. This normalized intertwining operator is entire and, for any fixed s_0, $M_n^*(s_0,\chi)$ is not identically zero [**17**].

Next, if $\beta = {}^t\beta \in M_n(F)$, recall that the generalized Whittaker functional $W_\beta(s)$ is defined on $I_n(s,\chi)$, for sufficiently large $\mathrm{Re}(s)$, by the integral

$$W_\beta(s)(\Phi)(g) = \int_{\mathrm{Sym}_n(F)} \Phi(w_n n(b)g,s)\,\psi(-tr(b\beta))\,db.$$

Karel [7] proved that if $\det\beta \neq 0$, then $W_\beta(s)$ has an entire analytic continuation and satisfies a functional equation

$$W_\beta(-s)\circ M_n(s,\chi) = \gamma_n(s,\chi)W_\beta(s).$$

The function $\gamma_n(s,\chi)$ was computed by Piatetski-Shapiro and Rallis [**16** , Proposition 2.2], [**17**]

$$\gamma_n(s,\chi) = \frac{a_n(s,\chi)}{b_n(-s,\chi^{-1})}\frac{L(-s+\frac{1}{2},\chi_\beta\chi^{-1})}{L(s+\frac{1}{2},\chi_\beta\chi)}\epsilon_n(s,\chi,\beta),$$

where χ_β is the quadratic character associated to the quadratic form β and $\epsilon_n(s,\chi,\beta)$ has the form $Bq^{B'\cdot s}$ for constants B and B'.

Combining these results, we conclude that

$$\begin{aligned} W_\beta(s) \circ M_n^*(-s,\chi^{-1}) \circ M_n^*(s,\chi) & \\ &= \frac{1}{a_n(-s,\chi^{-1})}\frac{1}{a_n(s,\chi)}\gamma_n(-s,\chi^{-1})\gamma_n(s,\chi)W_\beta(s) \\ &= \frac{1}{b_n(-s,\chi^{-1})b_n(s,\chi)}\epsilon_n(s,\chi,\beta)\epsilon_n(-s,\chi^{-1},\beta)W_\beta(s). \end{aligned}$$

For convenience, we set

$$\eta_n(s,\chi) = \frac{1}{b_n(-s,\chi^{-1})b_n(s,\chi)}\epsilon_n(s,\chi,\beta)\epsilon_n(-s,\chi^{-1},\beta).$$

LEMMA 2.5: *Assume that χ is normalized.*

(i) *If $\chi^2 \neq 1$, then $b_n(s,\chi) = 1$.*

(ii) *If $\chi^2 = 1$ but $\chi \neq 1$, then the poles of $b_n(s,\chi)$ are simple and occur at the points*

$$s \in -\frac{n+1}{2} + k + \frac{i\pi}{\log q}\mathbb{Z} \qquad 1 \leq k \leq \left[\frac{n}{2}\right].$$

(iii) *If $\chi = 1$, then the poles of $b_n(s,\chi)$ are simple and occur at the points*

$$s \in -\frac{n+1}{2} + k + \frac{i\pi}{\log q}\mathbb{Z} \qquad 1 \leq k \leq \left[\frac{n}{2}\right],$$

and the points

$$s \in -\frac{n+1}{2} + \frac{2\pi i}{\log q}\mathbb{Z}.$$

Note that this lemma determines all the zeroes of the constant of proportionality $\eta_n(s,\chi)$.

Finally, we will need a beautiful result of Waldspurger concerning contragredients of irreducible representations of $G = \mathrm{Sp}_n(F)$. Let

$$\delta = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix} \in \mathrm{GSp}_n(F),$$

and for any representation π of G let

$$\pi^\delta(g) = \pi(\delta^{-1}g\delta).$$

Also recall that $\tilde{\pi}$ denotes the contragredient of an admissible representation π. Then for any irreducible admissible representation π of G, [15, Chapter 4, II.1, Théorème p.91]

$$\pi^{\delta} \simeq \tilde{\pi}.$$

Moreover, since conjugation by δ preserves the parabolic subgroup P and fixes the character of P which defines $I_n(s,\chi)$, we have an isomorphism

$$A : I_n(s,\chi)^{\delta} \xrightarrow{\sim} I_n(s,\chi)$$

defined by

$$(A\Phi)(g,s) = \Phi(\delta^{-1}g\delta, s).$$

Combining these facts with Proposition 2.2, we obtain

THEOREM 2.6: *Assume that χ is normalized as explained in section 1.*

(i) *If $\chi^2 \neq 1$, then $I_n(s,\chi)$ is irreducible for all s.*

(ii) *If $\chi^2 = 1$ but $\chi \neq 1$, then $I_n(s,\chi)$ is irreducible whenever s does not lie in the set*

$$\{\ \pm(-\frac{n+1}{2}+k)+\frac{i\pi}{\log q}\mathbb{Z} \mid 1 \le k \le \left[\frac{n}{2}\right]\} \cup \frac{i\pi}{\log q}\mathbb{Z}.$$

(iii) *If $\chi = 1$, then $I_n(s,\chi)$ is irreducible whenever s does not lie in the set*

$$\{\ \pm(-\frac{n+1}{2}+k)+\frac{i\pi}{\log q}\mathbb{Z} \mid 1 \le k \le \left[\frac{n}{2}\right]\ \} \cup \{\ \pm\frac{n+1}{2}+\frac{2\pi i}{\log q}\mathbb{Z}\ \} \cup \frac{i\pi}{\log q}\mathbb{Z}.$$

Remark: This result is almost sharp. In section 5 below we will prove that when $\chi^2 = 1$ and $s \in \frac{i\pi}{\log q}\mathbb{Z}$ and n *is even*, then $I_n(s,\chi)$ is again irreducible. At all remaining points we will exhibit proper submodules associated to quadratic forms (section 3) and will determine the composition series (sections 4 and 5).

Proof: Suppose that $W \subset I_n(s,\chi)$ is an irreducible proper submodule and consider the short exact sequence

$$0 \longrightarrow W \longrightarrow I_n(s,\chi) \longrightarrow C \longrightarrow 0,$$

and its contragredient

$$0 \longrightarrow \tilde{C} \longrightarrow I_n(-s,\chi^{-1}) \longrightarrow \tilde{W} \longrightarrow 0.$$

Here we are using the fact that the contragredient of $I_n(s,\chi)$ is isomorphic to $I_n(-s,\chi^{-1})$. Using the result of Waldspurger, we then obtain a non-zero intertwining operator

$$T : I_n(-s,\chi^{-1}) \longrightarrow \tilde{W} \simeq W^\delta \hookrightarrow I_n(s,\chi)^\delta \simeq I_n(s,\chi).$$

Note that $\ker T = \tilde{C} \neq 0$. Since C is non-zero by our assumption that W is a proper submodule, we may repeat this argument beginning with any non-zero irreducible submodule

$$Z \subset \tilde{C}^\delta \subset I_n(-s,\chi^{-1})^\delta \simeq I_n(-s,\chi^{-1})$$

to obtain a non-zero intertwining operator

$$T' : I_n(s,\chi) \longrightarrow \tilde{Z} \simeq Z^\delta \subset \tilde{C} \subset I_n(-s,\chi^{-1}).$$

Note that $\ker T' \simeq \tilde{C}' \neq 0$, where C' is the quotient $I_n(-s,\chi^{-1})/Z$.

The induced representation $I_n(s,\chi)$ contains a subspace S of functions, supported in the open cell Pw_nN. This space is spanned by functions of the form

$$\Phi(w_n n(b), s) = \varphi(b)$$

for some $\varphi \in S(\mathrm{Sym}_n(F))$. For such a function $\Phi(s)$ we have

$$W_\beta(s)(\Phi)(e) = \int_{\mathrm{Sym}_n(F)} \varphi(b)\psi(-tr(b\beta))\,db = \hat{\varphi}(\beta).$$

In particular, this integral is independent of s, and for any given $\varphi \in S(\mathrm{Sym}_n(F))$, there exists a β with $\det\beta \neq 0$ for which $W_\beta(s)(\Phi)(e) = \hat{\varphi}(\beta) \neq 0$.

LEMMA 2.7: *Assume that*

$$[b_n(s,\chi)b_n(-s,\chi^{-1})]^{-1} \neq 0,$$

and hence that $\eta_n(s,\chi) \neq 0$. *Also suppose that if* $\chi^2 = 1$, *then* s *is not in the set* $\frac{i\pi}{\log q}\mathbb{Z}$. *Then* $M_n^*(s,\chi)$ *and* $M_n^*(-s,\chi^{-1})$ *are injective.*

Proof: As above, we consider the operator $M^*(-s,\chi^{-1})\circ M^*(s,\chi) : I_n(s,\chi) \longrightarrow I_n(s,\chi)$. By (ii) of Proposition 2.2 and our assumption on s and χ it follows that any intertwining map from $I_n(s,\chi)$ to itself must be a scalar. Applying the functional $W_\beta(s)$ to a suitable function in S, we conclude that this scalar must be $\eta_n(s,\chi)$. Thus $M_n^*(s,\chi)$ must be injective. Since our hypothesis is invariant under $s \mapsto -s$, the same argument shows that $M_n^*(-s,\chi^{-1})$ is also injective.

Now suppose that that s is such that $[b_n(s,\chi)b_n(-s,\chi^{-1})]^{-1} \neq 0$, and that, in the case $\chi^2 = 1$, s does not lie in the set $\frac{i\pi}{\log q}\mathbb{Z}$. Then either $\chi^2 \neq 1$ or $\chi^2 = 1$ and $q^s \neq \pm 1$. Thus (i) of Proposition 2.2 implies that at least one of the spaces $\mathrm{Hom}_G(I_n(s,\chi), I_n(-s,\chi^{-1}))$ and $\mathrm{Hom}_G(I_n(-s,\chi^{-1}), I_n(s,\chi))$ is one dimensional. Therefore either T is proportional to $M_n^*(-s,\chi^{-1})$ or T' is proportional to $M_n^*(s,\chi)$. This yields a contradiction, since neither T nor T' is injective when a proper irreducible submodule W exists. ∎

We must still prove irreducibility in the cases $s \in \frac{i\pi}{\log q}\mathbb{Z}$ in the cases $\chi^2 = 1$ but n even. This case is more delicate and will be handled in section 5 below.

3. Submodules associated to quadratic forms

When the character χ is quadratic and for certain values of s, the representations $I(s,\chi)$ have submodules associated to quadratic spaces. In this section we *do not* require χ to be normalized.

First, following §1.2 of [**13**], we recall a few facts about quadratic forms. For a non-degenerate inner product space V, $(\ ,\ )$ over F of even dimension m, let

$$\Delta(V) = (-1)^{m/2}\det(V) \in F^\times/F^{\times,2}$$

be the discriminant of V where $\det(V) = \det((x_i,x_j))$ for any basis $x_1,\cdots,x_m$ of V. For $x \in F^\times$, let

$$\chi_V(x) = (x,\Delta(V))_F,$$

where $(\cdot,\cdot)_F$ is the Hilbert symbol for F. The element $\Delta(V)$ in $F^\times/F^{\times,2}$ is determined by χ_V. The isometry class of V is then determined by m,χ_V, and the Hasse invariant $\epsilon(V)$, defined by

$$\epsilon(V) = \prod_{i<j}(a_i,a_j)_F$$

if we take any basis $\{x_i\}$ for V such that $(x_i,x_j) = \delta_{ij}a_i$ [**20**]. Note then that if χ and $m > 2$ are fixed, there are precisely two isometry classes of forms of dimension m with $\chi_V = \chi$, corresponding to the two possible choices of $\epsilon(V)$.

When $m = 2$ there are two forms when $\chi \neq 1$ and only one (the split form) when $\chi = 1$.

For a non-degenerate quadratic space V and for our fixed additive character ψ of F, let $(\omega_V, S(V^n))$ denote the Weil representation of G realized on $S(V^n)$, the space of Schwartz-Bruhat functions on V^n, in the usual Schrödinger model. The action of G commutes with the natural action of $O(V)$, and we will sometimes write $\omega_V(g,h)$ for the simultaneous operation of elements $g \in G$ and $h \in O(V)$.

Let $R_n(V)$ denote the image of the map

$$S(V^n) \to I_n(s_0, \chi_V)$$
$$\varphi \mapsto \Phi,$$

where

$$\Phi(g) = \omega_V(g)\varphi(0)$$

and $s_0 = \frac{m}{2} - \rho_n$. This map induces an isomorphism [18],

$$S(V^n)_{O(V)} \simeq R_n(V),$$

where $S(V^n)_{O(V)}$ is the space of $O(V)$-coinvariants.

A fact of fundamental importance for us is the following:

Proposition 3.1. *Assume that $m = \dim(V) \leq n$ so that $s_0 < 0$. Then $R_n(V)$ is an irreducible and unitarizable $G = G_n$ module. In fact, the restriction of this representation to P is also irreducible.*

Before giving the proof of this Proposition we recall a construction of Li [14]. For φ_1 and $\varphi_2 \in S(V^n)$, consider the pairing defined by

$$\begin{aligned}(\varphi_1, \varphi_2)_1 &= \int_{O(V)} (\varphi_1, \omega(h)\varphi_2)\, dh \\ &= \int_{O(V)} \int_{V^n} \varphi_1(x)\overline{\varphi_2(h^{-1}x)}\, dx\, dh.\end{aligned}$$

Since we are assuming that $\dim V = m \leq n$, the dual pair $(G, O(V))$ is in the stable range with $O(V)$ the small group. In this situation, Li proved that the above integral is absolutely convergent for all φ_1 and φ_2, and defines a positive semi-definite, G-invariant Hermitian form on $S(V^n)$. Let $R \subset S(V^n)$ be the radical of this pairing. Then Li also proved [14 ,§5.] that the quotient

$$H(1) = S(V^n)/R$$

is a non-zero irreducible unitarizable representation of G.

For fixed $\varphi_1 \in S(V^n)$, the map $\varphi \mapsto (\varphi, \varphi_1)_1$ defines an $O(V)$-invariant linear functional on $S(V^n)$, and thus factors through $R_n(V)$. In particular, there is a natural map

$$R_n(V) \longrightarrow H(1) = S(V^n)/R.$$

PROPOSITION 3.2: *When* $\dim V = m \leq n$, $R_n(V) \simeq H(1)$.

Proof: It will suffice to show that for any $\varphi \in R$, the associated function $\Phi(g) = \omega(g)\varphi(0)$ in $I_n(s_0, \chi_V)$ vanishes identically. In fact, since R is stable under the action of G, it will suffice to prove that

$$\hat{\varphi}(0) = \int_{V^n} \varphi(x)\, dx = 0$$

whenever $\varphi \in R$.

Now if $\varphi \in R$, then for any $\varphi_1 \in S(V^n)$, we have

$$\begin{aligned}(\varphi, \varphi_1) &= \int_{O(V)} \int_{V^n} \varphi(x)\overline{\varphi_1(h^{-1}x)}\, dx\, dh \\ &= \int_{V^n} \int_{O(V)} \varphi(hx)\, dh\, \overline{\varphi_1(x)}\, dx.\end{aligned}$$

Let μ be the moment mapping

$$\mu : V^n \to \mathrm{Sym}_n(F), \qquad x \mapsto (x, x) = ((x_i, x_j)),$$

where $x = (x_1, x_2, \ldots, x_n) \in V^n$. Then we let

$$V^n_{\rm reg} = (V^n)_{\rm reg} = \{x \in V^n \mid x \text{ and } \mu(x) \text{ have maximal rank }\}.$$

Here the rank of $x \in V^n$ means the dimension of the subspace of V spanned by the components of x, so that, for $x \in V^n_{\rm reg}$ this rank is equal to $\min(m, n) = m$, and is likewise equal to the rank of the $n \times n$ symmetric matrix (x, x). Let $\mathcal{O}_V = \mu(V^n_{\rm reg}) \subset \mathrm{Sym}_n(F)$ be the image of $V^n_{\rm reg}$. We then obtain a submersive map $V^n_{\rm reg} \longrightarrow \mathcal{O}_V$ and, by Harish-Chandra's result [4 , Theorem 11, p.49], a *surjective* map

$$S(V^n_{\rm reg}) \longrightarrow S(\mathcal{O}_V)$$

$$\varphi_1 \mapsto M_{\varphi_1}.$$

Taking $\varphi_1 \in S(V^n_{\text{reg}})$ above, and noting that for $x \in V^n_{\text{reg}}$,

$$H \cdot x = \mu^{-1}((x,x)),$$

we obtain

$$(\varphi, \varphi_1) = \int_{\mathcal{O}_V} F_\varphi(b)\, M_{\bar{\varphi}_1}(b)\, d_{\mathcal{O}_V} b,$$

where

$$F_\varphi(b) = \int_H \varphi(hx)\, dh,$$

for any choice of $x \in V^n_{\text{reg}}$ with $\mu(x) = b$. By the surjectivity of $\varphi_1 \mapsto M_{\varphi_1}$, we conclude that $F_\varphi \cong 0$ on $\mathcal{O}_V$. But now

$$\begin{aligned}\hat{\varphi}(0) &= \int_{V^n} \varphi(x)\, dx = \int_{V^n_{\text{reg}}} \varphi(x)\, dx \\ &= \int_{\mathcal{O}_V} F_\varphi(b)\, db = 0. \qquad \blacksquare\end{aligned}$$

Proposition 3.1 is now equivalent to Li's result on $H(1)$.

Remark 3.3: In fact, Li proves that $H(1)$ (and, indeed, the analogous space $H(\sigma)$ for an arbitrary irreducible unitary representation of $O(V)$) is irreducible when restricted to the maximal parabolic subgroup of G with Levi factor isomorphic to $\mathrm{GL}(m) \times \mathrm{Sp}(n-m)$. ∎

Next we consider the spaces $R_n(V)$ for different V's.

PROPOSITION 3.4:

(i) *Suppose that V_1 and V_2 are quadratic spaces of dimension m which are not isometric. Let $s_0 = \frac{m}{2} - \rho_n$. If $m \leq n+1$, then $R_n(V_1)$ and $R_n(V_2)$ are inequivalent representations of G_n.*

(ii) *If V is a quadratic space with $\dim(V) = m > 2n+2$, or $\dim(V) = m = 2n+2$ and $\chi_V \neq 1$, then, for $s_0 = \frac{m}{2} - \rho_n$,*

$$R_n(V) = I_n(s_0, \chi_V).$$

Observe that this assertion follows immediately from the irreducibility of $I_n(s_0, \chi)$ at this point.

(iii) *If V is a split quadratic space with $\dim(V) = m = 2n+2$ or $2n$, then*

$$R_n(V) = I_n(s_0, 1).$$

Proof: When χ is unramified, these facts are contained in the discussion, based on the results of [**3**], on pages 377-383 of [**18**]. We will give more or less complete proofs here in the general case.

For any quadratic space V of dimension m let $\mu : V^n \longrightarrow \mathrm{Sym}_n(F)$ be the moment map as above, and for $\beta \in \mathrm{Sym}_n(F)$ let $\Omega_\beta = \mu^{-1}(\beta)$ be the corresponding hyperboloid. It is a closed subset of V^n. Let ψ_β be the character of N given by $\psi_\beta(n(b)) = \psi(\frac{1}{2}tr(b\beta))$. The following fact is well known [**18**]:

LEMMA 3.5:

(i) *The twisted Jacquet functor $S(V^n) \longrightarrow S(V^n)_{N,\psi_\beta}$ can be explicitly realized as the restriction map $S(V^n) \longrightarrow S(\Omega_\beta)$.*

(ii) *If $\Omega_\beta = \emptyset$, then $S(V^n)_{N,\psi_\beta} = 0$.*

(iii) *If $\beta \in \mu(V^n_{\mathrm{reg}})$ where V^n_{reg} is as above, then Ω_β is a single $O(V)$ orbit, and the space*

$$R_n(V)_{N,\psi_\beta} \simeq \left(S(V^n)_{O(V)}\right)_{N,\psi_\beta} \simeq \left(S(V^n)_{N,\psi_\beta}\right)_{O(V)}$$

is one dimensional. The map $S(V^n) \longrightarrow R_n(V)_{N,\psi_\beta}$ is given by integration against an $O(V)$ invariant measure on Ω_β.

Now if V_1 and V_2 are as in the Proposition with $m \leq n$, the sets $\mu(V^n_{1,\mathrm{reg}})$ and $\mu(V^n_{2,\mathrm{reg}})$ are disjoint $\mathrm{GL}_n(F)$ orbits in $\mathrm{Sym}_n(F)$. Thus the representations $R_n(V_1)$ and $R_n(V_2)$ are distinguished by their twisted Jacquet spaces, by Lemma 3.5.

Next suppose that $m = n+1$, and note that $\beta \in \mathrm{Sym}_n(F)$ with $\det\beta \neq 0$ is represented by V, i.e., is in the image of the moment map, if and only if

$$\epsilon(V) = \epsilon(\beta)(-\det(V), \det\beta)_F.$$

LEMMA 3.6: *If $m = n+1$ and V_1 and V_2 are inequivalent quadratic spaces of dimension m, then there exist a $\beta \in \mathrm{Sym}_n(F)$ with $\det\beta \neq 0$ which is represented by V_1 but not by V_2.*

Proof: If $n \geq 3$, and $\epsilon(V_1) \neq \epsilon(V_2)$, we take β with $\det\beta = 1$ and with $\epsilon(\beta) = \epsilon(V_1)$. Then β is represented by V_1 but not by V_2. On the other hand, if $\epsilon(V_1) = \epsilon(V_2)$ and $\det(V_1) \neq \det(V_2)$, we take β with $\det\beta$ such that $(-\det(V_1), \det\beta)_F \neq (-\det(V_2), \det\beta)_F$ and with $\epsilon(\beta) = \epsilon(V_1)(-\det(V_1), \det\beta)_F$. Then, again, β is represented by V_1 but not by V_2. When $n = 1$, our assertion is well known and can be checked by a similar argument. ∎

The combination of Lemma 3.6 and Lemma 3.5 proves that $R_n(V_1)$ and $R_n(V_2)$ are inequivalent G_n modules. This finishes the proof of (i) of Proposition 3.4.

Next we prove (ii). If $\dim(V) = m \geq n$, let V^n_{sub} be the subset of V^n consisting of x whose rank, as defined above, is n. Note that $V^n_{\mathrm{reg}} \subset V^n_{\mathrm{sub}}$, and that V^n_{sub} is the subset of V^n on which the moment map μ is submersive.

Now assume that the restriction of the moment map μ to

$$\mu : V^n_{\mathrm{sub}} \longrightarrow \mathrm{Sym}_n(F)$$

is surjective, and recall that, when this is the case, the Weil orbital integral map [22]

$$S(V^n_{\mathrm{sub}}) \longrightarrow S(\mathrm{Sym}_n(F))$$
$$\varphi \mapsto M_\varphi$$

is *surjective.* In fact, by Theorem 11 of [4], the function M_φ is characterized by the fact that for any function $f \in S(\mathrm{Sym}_n(F))$,

$$\int_{V^n_{\mathrm{sub}}} f(\mu(x))\varphi(x)\,dx = \int_{\mathrm{Sym}_n(F)} f(y)M_\varphi(y)\,dy,$$

for our fixed Haar measures dx on V^n and dy on $\mathrm{Sym}_n(F)$. Since V^n_{sub} is open and dense in V^n, we also have

$$\begin{aligned}\omega_V(wn(b))\varphi(0) &= \gamma \int_{V^n} \psi(tr(b\mu(x))\varphi(x)\,dx \\ &= \gamma \int_{\mathrm{Sym}_n(F)} \psi(tr(b\cdot y))M_\varphi(y)\,dy \\ &= \gamma \hat{M}_\varphi(b).\end{aligned}$$

where $\hat{M}_\varphi$ is the Fourier transform of M_φ.

Thus, since any function in $S(\mathrm{Sym}_n(F))$ has the form $\hat{M}_\varphi$ for some choice of φ, $R_n(V)$ contains all functions in $I_n(s_0,\chi)$ which are supported on the open Bruhat cell, i.e., the space

$$I_n^{open}(s_0,\chi) = \{\Phi \in I_n(s_0,\chi) \mid \mathrm{support}(\Phi) \subset PwN\}.$$

It is clear that this space generates $I_n(s_0,\chi)$, as a G module. Thus $R_n(V) = I_n(s_0,\chi)$ whenever $m \geq n$ *and the surjectivity assumption holds.* But it is easy to check that the required surjectivity holds precisely when V has isotropic subspaces of dimension n, and that these occur for the V's described in (ii) and (iii) of Proposition 3.4. ∎

Combining this result with (i) of Proposition 2.4, we obtain:

COROLLARY 3.7: *Assume that n is odd and $\chi^2 = 1$. Here χ need not be normalized. Let V_1 and V_2 be the inequivalent quadratic spaces with* $\dim(V_1) = \dim(V_2) = n+1$ *and $\chi_{V_1} = \chi_{V_2} = \chi$. Here, if $n = 1$, assume that $\chi \neq 1$. Then $R_n(V_i)$ is an irreducible G module, and*

$$I_n(0,\chi) = R_n(V_1) \oplus R_n(V_2).$$

Proof: Since $I_n(0,\chi)$ is completely reducible, and we have in hand two subrepresentations $R_n(V_1)$ and $R_n(V_2)$, which are inequivalent by (i) of Proposition 3.4, we need only exclude the possibility $R_n(V_1) \subset R_n(V_2)$. But this is excluded by the fact that, via Lemmas 3.4 and 3.5, there exists a β for which $R_n(V_1)_{N,\psi_\beta} \neq 0$ but $R_n(V_2)_{N,\psi_\beta} = 0$. Recall that the twisted Jacquet functor is exact. ∎

Remark: Note that under the hypotheses of Corollary 3.7 but with $m \leq n$, we have a submodule

$$R_n(V_1) \oplus R_n(V_2) \subset I_n(s_0,\chi).$$

4. The N_1 Jacquet modules and exponents

We now let $P_1 \subset G$ be the parabolic subgroup which stabilizes an isotropic line, and chosen so that $P_1 \supset B$, our fixed Borel subgroup. Then $P_1 = M_1 N_1$ where

$$M_1 \simeq \mathrm{GL}_1(F) \times \mathrm{Sp}_{n-1}(F) = \mathrm{GL}_1(F) \times G_{n-1},$$

and

$$N_1 = \{\ m(\begin{pmatrix} 1 & u \\ 0 & 1_{n-1} \end{pmatrix})n(\begin{pmatrix} z & v \\ {}^tv & 0 \end{pmatrix}) \ \mid z \in F,\ u,\ v \in F^{n-1}\ \}.$$

We compute the Jacquet functor of $I_n(s,\chi)$ relative to N_1.

PROPOSITION 4.1: *As an $M_1 \simeq \mathrm{GL}_1 \times \mathrm{Sp}_{n-1}$ module, the space $I_n(s,\chi)_{N_1}$ has two composition factors:*

(i) *the quotient module $(\chi \cdot |\ |^{s+\rho_n}) \otimes I_{n-1}(s+\frac{1}{2},\chi)$, and*

(ii) *the submodule $(\chi^{-1} \cdot |\ |^{-s+\rho_n}) \otimes I_{n-1}(s-\frac{1}{2},\chi)$.*

More precisely, there is an exact sequence

$$0 \longrightarrow \chi^{-1}|\ |^{-s+\rho_n} \otimes I_{n-1}(s-\frac{1}{2},\chi) \xrightarrow{\alpha} I_n(s,\chi)_{N_1}$$
$$\xrightarrow{\beta} \chi|\ |^{s+\rho_n} \otimes I_{n-1}(s+\frac{1}{2},\chi) \longrightarrow 0.$$

Moreover, if $(\chi \otimes |\ |^s)^2 \neq 1$, *then this sequence splits and* $I_n(s,\chi)_{N_1}$ *is a direct sum of the spaces (i) and (ii).*

Proof: The proof is a standard calculation on Jacquet modules based simply on the fact that there are two elements in the double coset space $P_n\backslash \mathrm{Sp}_n/P_1$.

First, β is simply given by the restriction of functions to M_1. Next, to describe α, recall that [**12**]

$$G = P_nP_1 \coprod P_n w_1 P_1$$

with

$$w_1 = \begin{pmatrix} 0 & & 1 & \\ & 1_{n-1} & & \\ -1 & & 0 & \\ & & & 1_{n-1} \end{pmatrix},$$

and that w_1 commutes with the $\mathrm{Sp}(n-1)$ factor of M_1 and acts by inversion on the $\mathrm{GL}(1)$ factor. The kernel of the map β is the image in $I_n(s,\chi)_{N_1}$ of the space $T_n(s)$ of all $\Phi(s) \in I_n(s,\chi)$ which have support in the open cell $P_n w_1 P_1$. For such a function $\Phi(s)$, the map to the N_1-coinvariants may then be realized via the intertwining integral

$$U(s)\Phi(g) = \int_{U_1} \Phi(w_1ug,s)\,du$$

where

$$U_1 = \left\{ \begin{pmatrix} 1 & x & y & 0 \\ & 1_{n-1} & 0 & 0 \\ & & 1 & 0 \\ & & -{}^tx & 1_{n-1} \end{pmatrix} \right\}.$$

Here we take $g \in M_1$ (or even in $\mathrm{Sp}(n-1)$). Note that this function transforms by the character $|t|^{-s+\rho_n}$ of the $\mathrm{GL}(1)$ factor of M_1 [**12** , (1.2.9)]. Also note that, since the support of Φ is required to lie in the open cell $P_nw_1P_1$, this integral will be absolutely convergent for all s. In fact, since

$$P_n\backslash P_nw_1P_1 \simeq U_1 \times ((P_n \cap \mathrm{Sp}(n-1))\backslash \mathrm{Sp}(n-1)),$$

there is an isomorphism

$$T_n(s) \simeq S(U_1) \otimes I_{n-1}(s - \frac{1}{2}, \chi) \simeq S(F^{n-1}) \otimes S(F) \otimes I_{n-1}(s - \frac{1}{2}, \chi).$$

For $\varphi_1 \otimes \varphi_2 \otimes \varphi$ in the space on the right hand side, we set

$$\Phi(w_1 u(x,y)g, s) = \varphi_1(x)\varphi_2(y)\varphi(g)$$

and have

$$U(s)\Phi(g) = \hat{\varphi}_1(0)\hat{\varphi}_2(0)\varphi(g).$$

Thus $U(s)$ is surjective for all s, and it is easy to check that it induces an isomorphism

$$T(s)_{N_1} \xrightarrow{\sim} I_{n-1}(s - \frac{1}{2}, \chi).$$

It follows that the map $U(s)$ induces the inverse of α on the subspace $T(s)_{N_1}$ in $I_n(s_0, \chi)_{N_1}$.

The direct sum property follows from the disjointness of the characters $\chi \otimes |\ |^{s+\rho_n}$ and $\chi^{-1} \otimes |\ |^{-s+\rho_n}$ under the hypothesis of the Proposition. ∎

Next we calculate $R_n(V)_{N_1}$ for a quadratic space V. If V is isotropic, we let V' be the quadratic space obtained by deleting a hyperbolic plane form V. Note that V' is unique up to isomorphism, by Witt cancelation.

PROPOSITION 4.2: *Assume that $\chi^2 = 1$ and let V be a quadratic space with $\dim(V) = m$ and $\chi_V = \chi$. Let $s_0 = \frac{m}{2} - \rho_n$. Then*

(i) *As $M_1 \simeq \mathrm{GL}_1 \times \mathrm{Sp}_{n-1}$ modules, the sequence*

$$0 \longrightarrow \chi|\ |^{n+1-\frac{m}{2}} \otimes I_{n-1}(s_0 - \frac{1}{2}, \chi) \xrightarrow{\alpha} I_n(s_0, \chi)_{N_1} \xrightarrow{\beta} \chi|\ |^{\frac{m}{2}} \otimes I_{n-1}(s_0 + \frac{1}{2}, \chi) \longrightarrow 0$$

is exact.

(ii) *As $M_1 \simeq \mathrm{GL}_1 \times \mathrm{Sp}_{n-1}$ modules, the sequence*

$$0 \longrightarrow \chi|\ |^{n+1-\frac{m}{2}} \otimes R_{n-1}(V') \xrightarrow{\alpha'} R_n(V)_{N_1} \xrightarrow{\beta'} \chi|\ |^{\frac{m}{2}} \otimes R_{n-1}(V) \longrightarrow 0$$

is exact. Here, if V is isotropic, V' is the quadratic space of dimension $m-2$ defined above. If V is anisotropic, then $R_{n-1}(V')$ is taken to be zero.

(iii) *The natural maps between terms of the two sequences yield a commutative diagram*

$$\begin{array}{ccccc}
0 & \to & \chi|\,|^{n+1-\frac{m}{2}} \otimes I_{n-1}(s_0-\frac{1}{2},\chi) & \xrightarrow{\alpha} & I_n(s_0,\chi)_{N_1} \\
 & & i'' \uparrow & & i \uparrow \\
0 & \to & \chi|\,|^{n+1-\frac{m}{2}} \otimes R_{n-1}(V') & \xrightarrow{\alpha'} & R_n(V)_{N_1}
\end{array}$$

$$\begin{array}{cccc}
\xrightarrow{\beta} & \chi|\,|^{\frac{m}{2}} \otimes I_{n-1}(s_0+\frac{1}{2},\chi) & \to & 0 \\
 & \uparrow & & \\
\xrightarrow{\beta'} & \chi|\,|^{\frac{m}{2}} \otimes R_{n-1}(V) & \to & 0.
\end{array}$$

Here i is the natural map, and i'' is a non-zero multiple of the natural map.

Proof: Part (i) is just a special case of Proposition 4.1.

Next we begin to compute $R_n(V)_{N_1}$. Consider the exact sequence, induced by restriction of functions to the subspace of V^n where the first component is zero:

$$0 \longrightarrow X \longrightarrow S(V^n) \longrightarrow \chi|\,|^{\frac{m}{2}} \otimes S(V^{n-1}) \longrightarrow 0$$
$$\varphi \mapsto \varphi(0|\cdot).$$

Here X is the kernel of the restriction map, and we view these spaces as $P_1 \times O(V)$ modules. Taking N_1-coinvariants, we get an exact sequence

$$0 \longrightarrow X_{N_1} \longrightarrow S(V^n)_{N_1} \longrightarrow \chi|\,|^{\frac{m}{2}} \otimes S(V^{n-1}) \longrightarrow 0,$$

since N_1 acts trivially on the third term. At this point it is tempting to simply take the $O(V)$ co-invariants:

$$(X_{N_1})_{O(V)} \longrightarrow R_n(V)_{N_1} \longrightarrow \chi|\,|^{\frac{m}{2}} \otimes R_{n-1}(V) \longrightarrow 0,$$

but since this sequence is not necessarily left exact, we must proceed more carefully and give a more precise description of X_{N_1}.

Fix a non-zero isotropic vector $x_0 \in V$ and let $Q_1 \subset O(V)$ be the subgroup which stabilizes the isotropic line $F \cdot x_0 =< x_0 >$. Note that a Levi factor of Q_1 is isomorphic to $\mathrm{GL}(1) \times O(V')$ where

$$V' = x_0^{\perp}/ < x_0 > .$$

Also let Q_1^0 be the subgroup of Q_1 which fixes x_0.

Let $N_1^0 = N_1 \cap N$, and note that there is an isomorphism

$$S(V^n)_{N_1^0} \xrightarrow{\sim} S(\Omega_1)$$

given by restriction of functions to

$$\Omega_1 = \{ \ x = [x_1, x_2] \ \mid \ x_1 \in V, \ x_2 \in V^{n-1} \text{ with } (x_1, x_1) = 0, \ (x_1, x_2) = 0 \ \}.$$

Let Ω_1^0 be the open subset of $x \in \Omega_1$ for which $x_1 \neq 0$. Then there is an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_{N_1^0} & \longrightarrow & S(V^n)_{N_1^0} & \longrightarrow & \chi| \ |^{\frac{m}{2}} \otimes S(V^{n-1}) \longrightarrow 0 \\ & & \wr\uparrow & & \wr\uparrow & & \\ & & S(\Omega_1^0) & & S(\Omega_1) & & \end{array}$$

Note that, for our fixed isotropic vector x_0,

$$\Omega_1^0 = O(V) \cdot \{ \ [x_0, x] \mid x \in < x_0 >^{\perp} \ \}.$$

Now for $\varphi \in S(V^n)$, define a function φ' on $\mathrm{GL}(1) \times V'^{,n-1}$ by

$$\varphi'(t, x') = \int_{F^{n-1}} \omega(m(\begin{pmatrix} 1 & u \\ 0 & 1_{n-1} \end{pmatrix}))\varphi(tx_0, x)\, du,$$

where $x \in < x_0 >^{\perp}$ is any preimage of x'. Note that this function lies in $S(\mathrm{GL}(1)) \otimes S(V'^{,n-1})$ precisely when $\varphi \in X$. Finally, for $h \in O(V)$ and $\varphi \in X$, define

$$f_\varphi(h) = (\omega(h)\varphi)' \in S(\mathrm{GL}(1)) \otimes S(V'^{,n-1}).$$

These considerations yield the following result [8]

LEMMA 4.3: *As representations of $O(V) \times M_1$*

$$X_{N_1} \simeq \mathrm{Ind}_{Q_1 \times M_1}^{O(V) \times M_1}(\sigma)$$
$$\varphi \mapsto f_\varphi$$

where

$$\sigma = \xi \otimes \omega_1 \otimes \omega_{V'}^{(n-1)}$$

is the representation of $Q_1 \times M_1$ on the space $S(\mathrm{GL}(1)) \otimes S(V'^{,n-1})$ given as follows: σ is trivial on the unipotent radical of Q_1. The $\mathrm{GL}(1)$ in the Levi factor of Q_1 acts by the product of left translation on $S(\mathrm{GL}(1))$ with the character

$|\ |^{1-\frac{m}{2}}$. *The* GL(1) *in* M_1 *acts by the product of right translation on* $S(\mathrm{GL}(1))$ *with the character* $\chi|\ |^{n-1+\frac{m}{2}}$. *Finally, the group* $O(V') \times \mathrm{Sp}(n-1)$ *acts in the usual way, as a dual reductive pair, in the space* $S(V'^{,n-1})$.

Finally, we compute the $O(V)$ coinvariants in this induced representation. For this we use the following observation. Suppose that Q is any parabolic subgroup of $H = O(V)$ and that (σ, W) is a smooth representation of the Levi factor M of Q, viewed as a representation of Q, trivial on the unipotent radical of Q. Let $\mathrm{pr}_\sigma : W \longrightarrow W_{M,\delta_Q}$ be the natural projection of $\sigma \otimes \delta_Q^{\frac{1}{2}}$ to the maximal quotient on which M acts by the character δ_Q. Let

$$\phi \mapsto \oint_{Q\backslash H} \phi(h)\, d\mu(h),$$

be the H invariant linear functional, induced by some choice of a Haar measure on H, on the space of functions on H for which $\phi(qh) = \delta_Q(q)\phi(h)$ [1]. Here δ_Q is the modular function of Q.

LEMMA 4.4: *There is a natural isomorphism*

$$\mathrm{Ind}_Q^H(\sigma)_H \simeq \sigma_M$$
$$f \mapsto \oint_{Q\backslash H} \mathrm{pr}_\sigma(f).$$

We apply this in our case, with

$$W = S(\mathrm{GL}(1)) \otimes S(V'^{,n-1}).$$

Note that, via $\sigma \otimes \delta_{Q_1}^{\frac{1}{2}}$, the GL(1) in the Levi factor of Q_1 acts on functions in W simply by left multiplication on the GL(1) argument. We thus take

$$\mathrm{pr}_\sigma : S(\mathrm{GL}(1)) \otimes S(V'^{,n-1}) \longrightarrow R_{n-1}(V')$$

given by

$$\mathrm{pr}_\sigma(\varphi')(g) = \int_{F^\times} (\omega_{V'}^{(n-1)}(g)\varphi')(t,0)|t|^{m-2}\, d^\times t.$$

Note that for $t \in F^\times \simeq \mathrm{GL}(1)$ in the Levi factor of Q_1, $\delta_{Q_1}(t) = |t|^{m-2}$. This yields an isomorphism (the inverse of α'):

$$(X_{N_1})_{O(V)} \xrightarrow{\sim} \chi|\ |^{n+1-\frac{m}{2}} \otimes R_{n-1}(V'),$$

given by

$$\begin{aligned}
f_\varphi \mapsto &\oint_{Q_1\backslash O(V)} \mathrm{pr}_\sigma(f_\varphi)(h)\,d\mu(h)\\
&= \oint_{Q_1\backslash O(V)} \int_{F^\times} (\omega_{V'}^{(n-1)}(g,h)\varphi)'(t,0)|t|^{m-2}\,d^\times t\,d\mu(h)\\
&= \oint_{Q_1\backslash O(V)} \int_{F^\times} \int_{F^{n-1}} \omega(m(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}))(\omega_V^{(n)}(g,h)\varphi)(tx_0,0)|t|^{m-2}\,d^\times t\,d\mu(h)\\
&= \oint_{Q_1\backslash O(V)} \int_{F^\times} \int_{F^{n-1}} (\omega_V^{(n)}(g,h)\varphi)(tx_0, tx_0\cdot u)|t|^{m-2}\,d^\times t\,d\mu(h).
\end{aligned}$$

So far we have a commutative diagram:

$$\begin{array}{cccccccc}
0 & \to & \chi|\ |^{n+1-\frac{m}{2}}\otimes I_{n-1}(s_0-\frac12,\chi) & \xrightarrow{\alpha} & I_n(s_0,\chi)_{N_1} & & & \\
 & & & & i\uparrow & & & \\
 & & \chi|\ |^{n+1-\frac{m}{2}}\otimes R_{n-1}(V') & \xrightarrow{\alpha'} & R_n(V)_{N_1} & & & \\
 & & & & \xrightarrow{\beta} & \chi|\ |^{\frac{m}{2}}\otimes I_{n-1}(s_0+\frac12,\chi) & \to & 0\\
 & & & & & \uparrow & & \\
 & & & & \xrightarrow{\beta'} & \chi|\ |^{\frac{m}{2}}\otimes R_{n-1}(V) & \to & 0.
\end{array}$$

The map $i\circ\alpha'$ has image contained in the image of α, via the exactness of the right side of the diagram, and we thus obtain a map

$$\alpha^{-1}\circ i\circ\alpha' : \chi|\ |^{n+1-\frac{m}{2}}\otimes R_{n-1}(V')\simeq (X_{N_1})_{O(V)} \longrightarrow \chi|\ |^{n+1-\frac{m}{2}}\otimes I_{n-1}(s_0-\frac{1}{2},\chi).$$

We will now show that this map is a non-zero multiple of the natural map i' between these spaces. To do this, take $\varphi\in X$ and $g_1\in \mathrm{Sp}(n-1)$, and let

$$\Phi(g,s_0)=\omega(g)\varphi(0)$$

be the corresponding function in $I_n(s_0,\chi)$. According to the description of α given above, the corresponding function in $I_{n-1}(s_0+\frac12,\chi)$ is given by

$$\begin{aligned}
U(s_0)&\Phi(g_1)\\
&= \int_{U_1} \Phi(w_1u_1g_1,s_0)\,du_1\\
&= \int_{U_1} \omega(w_1u_1g_1)\varphi(0)\,du_1\\
&= \gamma_1\cdot\int_{U_1}\int_V \omega(u_1g_1)\varphi(v,0)\,dv\,du_1\\
&= \gamma_1\cdot\int_{F^{n-1}}\int_F\int_V \psi(\frac12 y(v,v))\omega(m(\begin{pmatrix}1 & x\\ & 1_{n-1}\end{pmatrix}))\omega(g_1)\varphi(v,0)\,dv\,dx\,dy.
\end{aligned}$$

Here γ_1 is the fourth root of unity which occurs in the action of w_1. For a moment we write φ in place of $\omega(m(\begin{pmatrix} 1 & x \\ & 1_{n-1} \end{pmatrix}))\omega(g_1)\varphi$, and consider the inner integral:

$$\begin{aligned}
&\int_F \int_V \psi(\frac{1}{2}y(v,v))\varphi(v,0)\,dv\,dy \\
=&\int_F \int_F \psi(yb)M_\varphi(b)\,db\,dy \\
=&\int_F \hat{M}_\varphi(y)\,dy \\
=&M_\varphi(0) \\
=&\int_{\mu^{-1}(0)} \varphi(v,0)\,d\mu(v) \\
=&\oint_{Q_1\backslash O(V)} \int_{F^\times} \omega(h)\varphi(tx_0,0)|t|^{m-2}\delta^\times t\,dh.
\end{aligned}$$

Here we have used the orbital integral map discussed above, in the case $n = 1$, and have taken a suitable normalization of the linear functional $\oint$. A more detailed discussion of the map $\varphi \mapsto M_\varphi$ in such a situation can be found in [**19** , §2]. Replacing φ with $\omega(m(\begin{pmatrix} 1 & x \\ & 1_{n-1} \end{pmatrix}))\omega(g_1)\varphi$ in this last formula, substituting the result into the outer integral above, and comparing the resulting expression with the image of f_φ in $R_{n-1}(V')$ computed earlier, we find that the map $\alpha^{-1} \circ i \circ \alpha'$ is γ_1^{-1} times the natural map, for the choice of measures we have made.

Since the natural map $i' : R_{n-1}(V') \longrightarrow I_{n-1}(s_0 + \frac{1}{2}, \chi)$ is injective [**18**], we conclude that α' must be injective as well. Moreover we obtain the comutative diagram of (iii) with $i'' = \gamma_1^{-1} \cdot i'$. This concludes the proof of (ii) and (iii). Note that (ii) is a special case of Theorem 2.8 of [8]. ∎

We may now use Propositions 4.1 and 4.2 to compute the exponents of the representations $I_n(s,\chi)$ and $R_n(V)$ along the Borel subgroup. More precisely, recall that U is the unipotent radical of our fixed Borel subgroup B, and that $B = AU = UA$ for the diagonal subgroup A as in section 1. For any admissible representation π of G the exponents of π along B are, by definition, the characters μ of A such that $a^{\mu+\rho}$ occurs in a generalized eigenspace decomposition of the Jacquet module π_U. Here $\rho = \rho_B = (n, n-1, \ldots, 1)$. Since U contains N_1, we may compute π_U in stages as $(\pi_{N_1})_{U\cap M_1}$.

Suppose that $(a_1, a_2, \ldots, a_r; b_1, \ldots, b_{n-r})$ is an ordered n-tuple of numbers which is divided into two subsets, the first r and the last $n-r$. We will call any permutation of this set which preserves the relative ordering of the subsets a *shuffle* of the set. Thus for a shuffle, a_1 will still come before a_2, etc. Note that the number of possible shuffles is $\binom{n}{r}$.

PROPOSITION 4.5: *The representation $I_n(s,\chi)$ has 2^n exponents which may be described as follows: For each r, with $0 \le r \le n$, every shuffle of the n-tuple*

$$(1-s-\rho_n, 2-s-\rho_n, \dots, r-s-\rho_n; s+\rho_n-n, \dots, s+\rho_n-r-1)$$

is an exponent. Moreover, these exponents are to be counted with multiplicity.

The exponents of $R_n(V)$ have a similar description:

PROPOSITION 4.6: *Let ℓ be the Witt index of V, i.e., the dimension of a maximal isotropic subspace of V. Then for each r, with $0 \le r \le \min(n,\ell)$, every shuffle of the n-tuple*

$$(1-\frac{m}{2}, 2-\frac{m}{2}, \dots, r-\frac{m}{2}; \frac{m}{2}-n, \dots, \frac{m}{2}-r-1)$$

is an exponent of $R_n(V)$, and these are all of the exponents. Again, these exponents are to be counted with multiplicity.

Proof: We simply must keep track of the sequence of characters of GL(1) which arises as we repeatedly apply either (i) or (ii) of Proposition 4.2. At each step there will be two choices. If we take a term involving $s_0 - \frac{1}{2}$ in (i) or one involving V' in (ii), we will say that we have moved 'left'; otherwise we will say that we have moved 'down'. A little inspection reveals that the exponent which arises on the first 'left' move, regardless of the number of intervening 'down's', will be $1-\frac{m}{2}$. Similarly, the second 'left' move produces $2-\frac{m}{2}$, etc. Likewise, the 'down' moves, as they occur, yield successively $\frac{m}{2}-n$, $\frac{m}{2}-n+1$, etc. The choice of the sequence of 'left' and 'down' moves yields an arbitrary shuffle. However, in the case of $R_n(V)$ the number of 'left' moves cannot exceed the Witt index ℓ of V, since each such move requires the removal of a hyperbolic plane from the part of V which remains at that step. ∎

5. Constituents and intertwining operators

With the hard work of section 4 completed, we may harvest some consequences.

First we take care of the two points where we have not yet proved the claimed irreducibility of $I_n(s,\chi)$.

PROPOSITION 5.1: *Suppose that n is even and that $\chi^2 = 1$. Then $I_n(s,\chi)$ is irreducible at the points $s = 0$ and $s = \frac{i\pi}{\log q}$, i.e., at all points $s \in \frac{i\pi}{\log q}\mathbb{Z}$.*

Proof: For convenience of notation, we consider the point $s = 0$ (the argument in the remaining case is identical) and suppose that $I_n(0,\chi)$ is not irreducible. Since we are on the unitary axis, Corollary 2.4 implies that $I_n(0,\chi) = W_1 \oplus W_2$ with W_1 and W_2 irreducible. This implies that the exact sequence of Proposition 4.1

$$0 \longrightarrow \chi|\ |^{\rho_n} \otimes I_{n-1}(-\frac{1}{2},\chi) \xrightarrow{\alpha} I_n(0,\chi)_{N_1} \xrightarrow{\beta} \chi|\ |^{\rho_n} \otimes I_{n-1}(\frac{1}{2},\chi) \longrightarrow 0,$$

splits. In fact, The image of W_{1,N_1} (say) under β must be non-zero, and hence, since $I_{n-1}(\frac{1}{2},\chi)$ is irreducible by Theorem 2.6, W_1 must map onto $I_{n-1}(\frac{1}{2},\chi)$. If this surjection has a non-zero kernel, the irreducibility of $I_{n-1}(-\frac{1}{2},\chi)$ (again by Theorem 2.6) would imply that $W_{1,N_1} = I_n(0,\chi)_{N_1}$. This in turn would force $W_{2,N_1} = 0$ which contradicts the faithfulness of the N_1-Jacquet functor. Thus the restriction of β to W_{1,N_1} must be an isomorphism and the sequence splits, as claimed.

But we have

LEMMA 5.2: *For n even and $\chi^2 = 1$, the sequence*

$$0 \longrightarrow \chi|\ |^{\rho_n} \otimes I_{n-1}(-\frac{1}{2},\chi) \xrightarrow{\alpha} I_n(0,\chi)_{N_1} \xrightarrow{\beta} \chi|\ |^{\rho_n} \otimes I_{n-1}(\frac{1}{2},\chi) \longrightarrow 0$$

does not split.

Proof: If the sequence were split, then the GL(1) in the Levi factor M_1 would act by the character $\chi|\ |^{\rho_n}$ in $I_n(0,\chi)_{N_1}$. To see that this is not the case, consider an arbitrary standard section $\Phi(s) \in I_n(s,\chi)$ and an element

$$t = t(a) = m(\begin{pmatrix} a & \\ & 1_{n-1} \end{pmatrix}) \in M_1.$$

Let $r_s(t)$ denote the action of t in the representation $I_n(s,\chi)$. Since t acts by the scalar $\chi(a)|a|^{\rho_n}$ in the quotient, $\chi|\ |^{\rho_n} \otimes I_{n-1}(\frac{1}{2},\chi)$, the image in $I_n(0,\chi)_{N_1}$ of the function

$$r_0(t)\Phi(0) - \chi(a)|a|^{\rho_n}\Phi(0)$$

lies in the kernel of β. Since the inverse of α is given by the integral $U(0)$, it will suffice to show that

$$U(0)\,[r_0(t)\Phi(0) - \chi(a)|a|^{\rho_n}\Phi(0)] \neq 0$$

for some choice of $\Phi(s)$. Note that the integral defining $U(0)$ only makes sense for the difference and may not converge when applied to the individual terms. On the other hand, for large $\mathrm{Re}(s)$, we may write

$$\begin{aligned} U(s)\left[r_s(t)\Phi(s) - \chi(a)|a|^{s+\rho_n}\Phi(s)\right] &= U(s)\left[r_s(t)\Phi(s)\right] - \chi(a)|a|^{s+\rho_n}U(s)\Phi(s) \\ &= \left[\chi(a)|a|^{-s+\rho_n} - \chi(a)|a|^{s+\rho_n}\right]U(s)\Phi(s), \end{aligned}$$

since $U(s)$ is then defined and intertwining on the whole space $I_n(s,\chi)_{N_1}$. The quantity $[\chi(a)|a|^{-s+\rho_n} - \chi(a)|a|^{s+\rho_n}]$ has a simple zero at $s = 0$. On the other hand, it is shown in [**12** , Proposition 1.2.4] that the intertwining operator $\mathbb{U}(s)$ which is defined by the integral $U(s)$ for sufficiently large $\mathrm{Re}(s)$ has a meromorphic analytic continuation and developes a *simple pole* at $s = 0$ when n is even and $\chi^2 = 1$. There exists a standard section $\Phi(s)$ for which the residue of $\mathbb{U}(s)\Phi(s)$ is non-zero at $s = 0$, and we then conclude that

$$\begin{aligned} U(0)\cdot\Big[\, r_0(t)\Phi(0) - \chi(a)|a|^{\rho_n}\Phi(0)\,\Big] &\\ &= \left(U(s)\cdot\Big[\, r_s(t)\Phi(s) - \chi(a)|a|^{s+\rho_n}\Phi(s)\,\Big]\right)\Big|_{s=0} \\ &= \left(\mathbb{U}(s)\cdot\Big[\, r_s(t)\Phi(s) - \chi(a)|a|^{s+\rho_n}\Phi(s)\,\Big]\right)\Big|_{s=0} \\ &= \left(\mathbb{U}(s)\cdot(r_s(t)\Phi(s)) - \chi(a)|a|^{s+\rho_n}\mathbb{U}(s)\cdot\Phi(s)\right)\Big|_{s=0} \\ &= \left(\chi(a)\left(|a|^{-s+\rho_n} - |a|^{s+\rho_n}\right)\mathbb{U}(s)\Phi(s)\right)\Big|_{s=0} \\ &= \chi(a)|a|^{\rho_n}(-2\log|a|)\cdot\operatorname*{Res}_{s=0}\big(\mathbb{U}(s)\Phi(s)\big) \\ &\neq 0. \end{aligned}$$

Thus t does *not* act by a scalar in $I_n(0,\chi)_{N_1}$ and our sequence is not split. ∎

In particular, the representation $I_n(0,\chi)$ must be irreducible and Proposition 5.1 is proved. ∎

We now turn to the points of reducibility and determine the disposition of the submodules $R_n(V)$.

PROPOSITION 5.3: *Suppose that V_1 and V_2 are inequivalent forms such that $\dim(V_i) = m$ and $\chi_{V_1} = \chi_{V_2} = \chi$, and suppose that $m \geq n+1$. Then, for $s_0 = \frac{m}{2} - \rho_n$,*

$$I_n(s_0, \chi) = R_n(V_1) + R_n(V_2).$$

Proof: First observe that when $s_0 = 0$ or when $s_0 \geq \rho_n$, our assertion is already contained in Corollary 3.7 or in (ii) and (iii) of Proposition 4.2. In particular, our claim is proved when $n = 1$. In general, since the N_1 Jacquet functor is exact and non-zero on all constituents of $I_n(s, \chi)$, it suffices to prove that

$$I_n(s_0, \chi)_{N_1} = (R_n(V_1) + R_n(V_2))_{N_1},$$

and this follows by induction on n, using (iii) of Proposition 4.2, remembering to shift the sequence by $-\rho_B$. Note that, in this induction, we need only consider $s_0 > 0$, so that the term involving $s_0 - \frac{1}{2}$ is always covered by the induction hypothesis. ∎

In the discussion which follows we will need a few more conventions. For any quadratic space V with $n + 1 \leq m = \dim(V) \leq 2n + 2$, let V_0 denote the 'complementary' quadratic space of dimension $2n+2-m$, if it exists. This space is determined by the condition that $V + (-V_0)$ is the split space of dimension $2n + 2$, where $-V_0$ denotes the space V_0 with the negative of the original inner product. Note that, if V_{an} is a maximal anisotropic subspace of V (this is unique up to isometry), then both V and V_0 may be obtained by adding split spaces of suitable dimensions to V_{an}. In the extreme case in which V is a split space of dimension $2n + 2$, we have $V_0 = 0$. No such complementary V_0 exists in the following cases:

(i) $\dim V = 2n + 2$ and $\chi \neq 1$ or $\chi = 1$ and V is quaternionic, i.e., V is the orthogonal sum of a quaternion norm form with a split space of dimension $2n - 2$.

(ii) $\dim V = 2n$ and V is quaternionic.

Whenever a space V and its complement V_0 are refered to in what follows, we implicitly assume that these possiblities (i) and (ii) for V are excluded.

Recall that there is a non-degenerate sesquilinear pairing

$$I_n(s, \chi) \otimes I_n(-\bar{s}, \chi) \longrightarrow \mathbb{C}$$

given by

$$< \Phi_1(s) \mid \Phi_2(-\bar{s}) >= \int_{\mathrm{Sym}_n(F)} \Phi_1(wn(b),s)\overline{\Phi_2(wn(b),-\bar{s})}\, db$$

for $\Phi_1(s) \in I_n(s,\chi)$ and $\Phi_2(-\bar{s}) \in I_n(-\bar{s},\chi)$. For V and V_0 complementary, as above, the restriction of this pairing to

$$R_n(V) \otimes R_n(V_0) \longrightarrow \mathbb{C}$$

is nonzero [**18** , p.369 and p.380], while, if U is any quadratic space of dimension $2n+2-m$ with $\chi_U = \chi$ which is not complementary to V, then

$$R_n(U) \subset R_n(V)^{\perp}.$$

Thus we have

LEMMA 5.4: *Suppose that V is a quadratic space with $n+1 \leq m = \dim(V) \leq 2n+2$, and let V_0 be the complementary space. Then*

$$\dim \mathrm{Hom}_G(R_n(V), \overline{R_n(V_0)^{\check{}}}) \neq 0.$$

Moreover, in this case, $R_n(V_0)$ is unitarizable, and so

$$\dim \mathrm{Hom}_G(R_n(V), R_n(V_0)) \neq 0.$$

Proof: First note that, for any quadratic space V, the conjugate $\overline{\omega_{\psi,V}}$ of the Weil representation $\omega_{\psi,V}$ of $G = \mathrm{Sp}_n(F)$ associated to V is equivalent to the Weil representation $\omega_{\psi,-V}$, associated to $-V$. Therefore, the non-triviality of our sesquilinear pairing on $R_n(V) \otimes R_n(V_0)$ follows from the discussion on p.380 of [**18**]. The unitarizability of $R_n(V_0)$, given by Proposition 3.1 above, implies that

$$\overline{R_n(V_0)^{\check{}}} \simeq R_n(V_0),$$

and immediately yields the last assertion. ∎

These facts will be useful in a moment.

PROPOSITION 5.5: *Let V_1 and V_2, etc. be as in Proposition 5.3, with $n+1 \leq m \leq 2n+2$, so that $0 \leq s_0 \leq \rho_n$. Assume that complementary quadratic spaces $V_{1,0}$ and $V_{2,0}$ exist. Then*

$$M_n^*(s_0)\big(R_n(V_i)\big) = R_n(V_{i,0})$$

and

$$M_n^*(s_0)\big(I_n(s_0,\chi)\big) = R_n(V_{1,0}) \oplus R_n(V_{2,0}).$$

Moreover, if $m = 2n+2$ or $2n$ and $\chi = 1$, let V_1 be the split form and let V_2 be the quaternionic form of dimension m. Then

$$M_n^*(s_0)(I_n(s_0,1)) = M_n^*(s_0)(R_n(V_1)) = R_n(V_{1,0}).$$

Proof: At first we exclude the cases in which a complement fails to exist for one of the spaces V_i, and we also exclude the case $s_0 = 0$. Then, by Proposition 4.4 of [**10**], we have

$$\dim \mathrm{Hom}_G(R_n(V_i), I_n(-s_0,\chi)) \leq 1.$$

On the other hand, any intertwining map from $R_n(V_i)$ to $R_n(V_{i,0})$ yields an intertwining map from $R_n(V_i)$ to $I_n(-s_0,\chi)$. Note that, since $R_n(V_{i,0})$ is irreducible, any non-zero intertwining map from $R_n(V_i)$ to this space must be surjective. Thus, by Lemma 5.4, we must have

$$\dim \mathrm{Hom}_G(R_n(V_i), R_n(V_{i,0})) = 1,$$

and the restriction of $M_n^*(s_0)$ to $R_n(V_i)$ defines an element of this space. If this element is non-zero, then its image is precisely $R_n(V_{i,0})$.

Thus it suffices to show that $M_n^*(s_0)$ is non-zero on the space $R_n(V_i)$. Since $M_n^*(s_0)$ is not identically zero, it must be non-trivial on at least one $R_n(V_i)$, say $R_n(V_1)$, by Proposition 5.3. Thus we obtain $M_n^*(s_0)\big(R_n(V_1)\big) = R_n(V_{1,0})$.

If $\chi \neq 1$, choose an element $a \in F^\times$ such that $\chi(a) \neq 1$. Then the space V_2 may be taken to be the space V_1 with the inner product scaled by the factor a, i.e., we may take $V_2 = a \cdot V_1$, in an unfortunate but temporary notation. Let

$$\tau_a = \begin{pmatrix} 1_n & \\ & a\cdot 1_n \end{pmatrix} \in \mathrm{GSp}_n(F)$$

and let ω_{ψ,V_1}^a be the conjugate of the Weil representation ω_{ψ,V_1} by the outer automorphism Ad τ_a of $G = \mathrm{Sp}_n(F)$. Then

$$\omega_{\psi,V_1}^a \simeq \omega_{\psi,a\cdot V_1} \simeq \omega_{\psi,V_2}.$$

Using this and the fact that Ad τ_a preserves $I_n(s,\chi)$, viewed as a space of functions on G, it is not difficult to check that $R_n(V_1)$ and $R_n(V_2)$ are switched by

Ad τ_a and similarly for $R_n(V_{1,0})$ and $R_n(V_{2,0})$. Finally, since

$$\int_{\mathrm{Sym}_n(F)} \Phi(\tau_a^{-1} wn(x) g\tau_a, s)\, dx = \int_{\mathrm{Sym}_n(F)} \Phi(\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix} wn(ax)\tau_a^{-1} g\tau_a, s)\, dx$$
$$= \chi(a)|a|^{n(s+\rho_n)-n\rho_n} M_n(s)\Phi(\tau_a^{-1} g\tau_a),$$

we have

$$M_n^*(s_0)\big(\operatorname{Ad} \tau_a \cdot \Phi\big) = \chi(a)|a|^{ns_0} \cdot \operatorname{Ad} \tau_a\big(M_n^*(s_0)\Phi\big).$$

Thus we obtain the required assertion $M_n^*(s_0)\big(R_n(V_2)\big) = R_n(V_{2,0})$ by applying Ad τ_a to the corresponding assertion for $R_n(V_1)$.

Next suppose that $\chi = 1$. In this case we know that the $K = \mathrm{Sp}_n(\mathcal{O})$ types in $\mathrm{Ind}_{P\cap K}^{K}(1)$ occur with multiplicity one [**17**]. This implies that the restriction of the normalized intertwining operator $M_n^*(s)$ to any K type θ in $I_n(s,1)$ is a scalar operator $A_\theta(s) = c_\theta(s)\cdot \mathrm{Id}_\theta$ where $c_\theta(s)$ is an entire function of s. We note that the property

$$M_n^*(-s)\circ M_n^*(s) = [b_n(-s)b_n(s)]^{-1}\cdot \mathrm{Id}$$

implies that

$$c_\theta(-s)c_\theta(s) = [b_n(-s)b_n(s)]^{-1}$$

for all θ. But then, if s_0 is as in our Proposition (and $s_0 \neq 0$!!), $b_n(s)^{-1}$ *vanishes* at $s = -s_0$ and is *nonzero* at $s = s_0$. Hence either $c_\theta(s)$ or $c_\theta(-s)$ (but not both) admits a zero at s_0! On the other hand, (i) of Proposition 4.4 of [**10**] implies that

$$\dim \mathrm{Hom}_G(R_n(V_{i,0}), I_n(s_0,1)) = 0.$$

Thus any θ which occurs in $R_n(V_{i,0})$ lies in the kernel of $M_n^*(-s_0)$ and hence has $c_\theta(-s_0) = 0$ and $c_\theta(s_0) \neq 0$. Thus $R_n(V_{1,0}) \oplus R_n(V_{2,0})$ lies in the image of $M_n^*(s_0)$. If $R_n(V_2)$ were in the kernel of $M_n^*(s_0)$ we would have

$$\begin{aligned} M_n^*(s_0)\big(I_n(s_0,1)\big) &= M_n^*(s_0)\big(R_n(V_1)+R_n(V_2)\big)\\ &= M_n^*(s_0)(R_n(V_1))\\ &= R_n(V_{1,0}).\end{aligned}$$

Thus $R_n(V_2)$ cannot lie in the kernel, unless $R_n(V_{2,0}) = 0$. But, when the complement $V_{2,0}$ exists, the space $R_n(V_{2,0})$ is non-zero. This proves that $R_n(V_2)$ is not in the kernel of $M_n^*(s_0)$ and yields our assertion.

Next we suppose that $m = n+1$ so that $s_0 = 0$. We already know that $I_n(0,\chi) = R_n(V_1) \oplus R_n(V_2)$ and that the $R_n(V_i)$'s are irreducible and inequivalent. Thus $M_n^*(0)$ must be an isomorphism on at least one of the $R_n(V_i)$'s. If $\chi \neq 1$, we may interchange the two constituents, as before. If $\chi = 1$, then we again consider the $c_\theta(s)$'s. Note that

$$b_n(s,1) = \zeta(s+\rho_n) \prod_{k=1}^{[\frac{n}{2}]} \zeta(2s+n+1-2k),$$

with $\zeta(s) = (1-q^{-s})^{-1}$, is holomorphic at $s=0$. Thus

$$c_\theta(0)^2 = b_n(0,1)^{-2} \neq 0,$$

and so $M_n^*(0)$ must be an isomorphism.

We must still check our assertions in the case $m = 2n+2$, $2n$ and $\chi = 1$. This will be done in section 6 below. ∎

Remark: By (i) of Proposition 2.2, we see that a basis for the intertwining operators from $I_n(s_0,\chi)$ to $I_n(-s_0,\chi)$ is given by the composition of $M_n^*(s_0)$ with the projections onto the two summands of $R_n(V_{1,0}) \oplus R_n(V_{2,0})$. Moreover, it follows from that same Proposition that

$$\dim \operatorname{Hom}_G(I_n(s_0,\chi), R_n(V_{i,0})) = 1$$

for $i = 1$, and 2. ∎

Next we determine the kernel of $M_n^*(s_0)$.

PROPOSITION 5.6: *With the same hypotheses as in Propositions 5.3 and 5.5 with $n+1 < m < 2n+2$, or for $m = 2n+2$ with $\chi = 1$*

$$\ker(M_n^*(s_0)) = R_n(V_1) \cap R_n(V_2).$$

If $m = 2n+2$, and $\chi \neq 1$, then $\ker(M_n^*(\rho_n)) = 0$. Finally, if $m = n+1$ then $\ker(M_n^*(0)) = 0$.

Proof: We will prove this by induction on n using Proposition 4.2. We note first, however, that for n odd, $M_n^*(0)$ is an isomorphism since, by the previous result its image is $R_n(V_1) \oplus R_n(V_2) = I_n(0,\chi)$. The case $m = 2n+2$ and $\chi \neq 1$

follows from the irreducibility of $I_n(\pm\rho_n, \chi)$, while the case $m = 2n+2$ with $\chi = 1$ will be taken care of in section 6 below.

We must then consider the case $n+1 < m < 2n+2$, and we let $Y_n(s_0) = \ker(M_n^*(s_0))$. For convenience we will temporarily drop χ from the notation. Applying the N_1 Jacquet functor to the sequence

$$0 \longrightarrow Y_n(s_0) \longrightarrow I_n(s_0) \xrightarrow{\lambda} R_n(V_{1,0}) \oplus R_n(V_{2,0}) \longrightarrow 0,$$

where λ is the map induced by $M_n^*(s_0)$, and using Proposition 4.2, we obtain:

$$\begin{aligned} 0 \longrightarrow Y_n(s_0)_{N_1} &\longrightarrow I_{n-1}(s_0 - \frac{1}{2}) \oplus I_{n-1}(s_0 + \frac{1}{2}) \\ &\xrightarrow{\lambda_1 \oplus \lambda_2} \big(R_{n-1}(V'_{1,0}) \oplus R_{n-1}(V'_{2,0})\big) \oplus \big(R_{n-1}(V_{1,0}) \oplus R_{n-1}(V_{2,0})\big) \longrightarrow 0. \end{aligned}$$

Here

$$\lambda_1 : I_{n-1}(s_0 - \frac{1}{2}) \longrightarrow R_{n-1}(V'_{1,0}) \oplus R_{n-1}(V'_{2,0}) \longrightarrow 0$$

and

$$\lambda_2 : I_{n-1}(s_0 + \frac{1}{2}) \longrightarrow R_{n-1}(V_{1,0}) \oplus R_{n-1}(V_{2,0}) \longrightarrow 0.$$

By the remark following Proposition 5.5, we see that λ_1 must have the same kernel as $M_{n-1}^*(s_0 - \frac{1}{2})$ and λ_2 must have the same kernel as $M_{n-1}^*(s_0 + \frac{1}{2})$. By induction, we have

$$\ker(\lambda_1) = R_{n-1}(V_1') \cap R_{n-1}(V_2')$$

and

$$\ker(\lambda_2) = R_{n-1}(V_1) \cap R_{n-1}(V_2).$$

Thus

$$Y_n(s_0)_{N_1} \simeq \big(R_{n-1}(V_1') \cap R_{n-1}(V_2')\big) \oplus \big(R_{n-1}(V_1) \cap R_{n-1}(V_2)\big).$$

On the other hand, we have

$$\big(R_n(V_1) \cap R_n(V_2)\big)_{N_1} \subset \big(R_n(V_1)\big)_{N_1} \cap \big(R_n(V_2)\big)_{N_1},$$

and, by (ii) and (iii) of Proposition 4.2,

$$\big(R_n(V_1)\big)_{N_1} \cap \big(R_n(V_2)\big)_{N_1} \simeq \big(R_{n-1}(V_1') \cap R_{n-1}(V_2')\big) \oplus \big(R_{n-1}(V_1) \cap R_{n-1}(V_2)\big).$$

Finally, we note that, by Proposition 5.3,

$$R_n(V_1)/(R_n(V_1)\cap R_n(V_2)) \simeq I_n(s_0,\chi)/R_n(V_2).$$

Applying the N_1 Jacquet functor to this it is easy to see that, in fact,

$$\big(R_n(V_1)\cap R_n(V_2)\big)_{N_1} = \big(R_n(V_1)\big)_{N_1}\cap\big(R_n(V_2)\big)_{N_1},$$

and this proves that

$$\big(R_n(V_1)\cap R_n(V_2)\big)_{N_1} = Y_n(s_0)_{N_1}.$$

Thus

$$R_n(V_1)\cap R_n(V_2) = Y_n(s_0)$$

by the non-triviality of the Jacquet functor on constituents. ∎

6. Exponents and composition series

In order to complete our description of the composition series of the representation $I_n(s,\chi)$ at points of reducibility, we will now make a more detailed study of the exponents and their multiplicities.

Recall that, by Proposition 4.5, for $s_0 = \frac{m}{2} - \frac{n+1}{2}$ the exponents of $I_n(s_0,\chi)$ are obtained as shuffles of sequences

$$(1-\frac{m}{2}, 2-\frac{m}{2},\ldots,r-\frac{m}{2}; \frac{m}{2}-n,\ldots,\frac{m}{2}-r-1)$$

where $0\le r\le n$. For convenience we will denote this exponent by $E_r = E_r(m)$ and will let

$$A_r = (1-\frac{m}{2}, 2-\frac{m}{2},\ldots,r-\frac{m}{2}),$$

and

$$B_r = (\frac{m}{2}-n,\ldots,\frac{m}{2}-r-1)$$

denote the first and second blocks in E_r. Thus $E_r = (A_r; B_r)$. We will omit the m unless more than one value is being considered. Recall also that a shuffle of E_r is any permutation in which the ordering of elements of A_r is preserved and the ordering of elements of B_r is preserved.

Now if V with $\dim V = m$ is a quadratic space for which $R_n(V)\subset I_n(s_0,\chi)$, the exponents of $R_n(V)$ are obtained as shuffles of E_r's for r's which do not exceed the Witt index (dimension of a maximal isotropic subspace) of V.

The main case of interest to us will be $\chi \neq 1$ and $n+1 < m \leq 2n$. In this case, there will be two spaces V_1 and V_2 of dimension m, of opposite Hasse invariants, and both of Witt index $\frac{m}{2} - 1$. Let $V_{i,0}$ be the complementary space to V_i as in section 5 above, and recall that $\dim V_i = m'$ is determined by the condition $m + m' = 2n + 2$. We have shown that

$$I_n(s_0, \chi) = R_n(V_1) + R_n(V_2)$$

and that $R_n(V_{1,0}) \oplus R_n(V_{2,0})$ is a submodule of $I_n(-s_0, \chi)$. Moreover, since $\chi \neq 1$, we have seen that there is an outer automorphism of G which preserves the representation $I_n(s_0, \chi)$ (resp. $I_n(-s_0, \chi)$) and interchanges the submodules $R_n(V_1)$ and $R_n(V_2)$ (resp. $R_n(V_{1,0})$ and $R_n(V_{2,0})$). Since this outer automorphism preserves the Borel subgroup, our fixed maximal split torus, etc., it preserves exponents. Thus the submodules $R_n(V_1)$ and $R_n(V_2)$ (resp. $R_n(V_{1,0})$ and $R_n(V_{2,0})$) have the same set of exponents.

LEMMA 6.1: *The set of exponents of $I_n(s_0, \chi)$ which occur as shuffles of $E_r(m)$'s with $r \geq \frac{m}{2}$ is identical to the set of exponents of $R_n(V_{i,0})$.*

Proof: The exponents of $R_n(V_{1,0})$ are shuffles of

$$E_{r'}(m') = (1 - \frac{m'}{2}, 2 - \frac{m'}{2}, \ldots, r' - \frac{m'}{2}; \frac{m'}{2} - n, \ldots, \frac{m'}{2} - r' - 1)$$

for $0 \leq r' \leq \frac{m'}{2} - 1$. But now observe that

$$1 - \frac{m'}{2} = \frac{m}{2} - n \qquad \text{and} \qquad \frac{m'}{2} - n = 1 - \frac{m}{2}.$$

Also, setting $r = n - r'$, so that $\frac{m}{2} \leq r \leq n$, we have

$$r' - \frac{m'}{2} = \frac{m}{2} - r - 1 \qquad \text{and} \qquad \frac{m}{2} - r - 1 = r' - \frac{m'}{2}.$$

Thus $A_{r'}(m') = B_r(m)$ and $B_{r'}(m') = A_r(m)$, and the set of shuffles of $E_{r'}(m')$ coincides with the set of shuffles of $E_r(m)$. This gives the required identification of exponents. ∎

Note that the set of exponents here are the ones whose full multiplicity does not occur in $R_n(V_i)$. Also note that since $I_n(s_0, \chi) = R_n(V_1) + R_n(V_2)$, the exponents which occur outside of $R_n(V_1)$ must occur in $R_n(V_2)$, and hence, since $R_n(V_1)$ and $R_n(V_2)$ have the same exponents, must be repeated inside of $R_n(V_1)$.

Our goal now is to investigate the multiplicities of the exponents. To do this it is useful to formulate the combinatorial problem involved more abstractly.

Suppose that m and n are given positive integers with m even and with $n+1 < m < 2n$. Let R be the set of sequences (counted with multiplicities) which arise as shuffles of sequences E_r defined as above, with $0 \le r \le \frac{m}{2} - 1$. Similarly, let S be the set of sequences (counted with multiplicities) which arise as shuffles of sequences E_s for $\frac{m}{2} \le s \le n$. Finally, let $I = R \cup S$, again counting multiplicities.

PROPOSITION 6.2:

(i) $S \subset R$, *i.e., the shuffles of* E_s *all occur as shuffles of* E_r

(ii) $I - 2S = R - S$ *is of multiplicity* 1. *Moreover,*

$$(I - 2S) \cap S = (R - S) \cap S = \emptyset.$$

Thus the shuffles which remain after the overlap of R *and* S *is removed are all distinct and do not occur in the set* S.

Proof: First observe:

LEMMA 6.3: *A shuffle of* E_r *and a shuffle of* E_s *can coincide only if either* $r = s$ *or* $r + s = m - 1$.

Proof: Since each block in E_r is strictly increasing, the largest component of E_r is $\max(r - \frac{m}{2}, \frac{m}{2} - r - 1)$. If a shuffle of E_r coincides with a shuffle of E_s, then we must have, in particular,

$$\max(r - \frac{m}{2}, \frac{m}{2} - r - 1) = \max(s - \frac{m}{2}, \frac{m}{2} - s - 1),$$

and hence either $r = s$ or $r + s = m - 1$. ∎

Thus we may as well fix r and s with $r + s = m - 1$ and

$$0 \le r \le \frac{m}{2} - 1 < \frac{m}{2} \le s \le n.$$

Let

$$\begin{aligned}\alpha &= (1 - \frac{m}{2}, 2 - \frac{m}{2}, \dots, \frac{m}{2} - n - 1),\\ \beta &= (\frac{m}{2} - n, \dots, r - \frac{m}{2})\\ &= (\frac{m}{2} - n, \dots, \frac{m}{2} - s - 1)\end{aligned}$$

and

$$\gamma = (r+1-\frac{m}{2},\ldots,\frac{m}{2}-r-1).$$

Here each sequence increases by 1 in each step. Then, because of the conditions we have imposed on r, s, m and n, we may write

$$E_r = (\alpha, \beta_1; \beta_2, \gamma)$$

and

$$E_s = (\alpha, \beta', \gamma; \beta'').$$

Here in the expression for E_r (resp. E_s) we write β_1 and β_2 (resp. β' and β'') to distinguish the two copies of β. Now we claim that any shuffle of E_s can be obtained as a shuffle of E_r, and that the procedure for obtaining this shuffle from that of E_s will yield distinct results of distinct shuffles of E_s (here we keep track of the actual shuffle and not just of the sequence which it creates).

For an arbitrary shuffle of E_s, let $L(\beta'')$ be the initial segment of β'' consisting of elements which are moved to the left of some element of α, and let $R(\beta'')$ be the terminal segment of β'' consisting of elements which are moved to the right of some element of γ. Then write

$$\beta'' = (L(\beta''), M(\beta''), R(\beta'')),$$

where $M(\beta'')$ is what remains. Note that $M(\beta'')$ might be empty. The given shuffle of E_s can be written as

$$(Sh(\alpha; L(\beta'')), Sh(\beta'; M(\beta'')), Sh(\gamma; R(\beta''))),$$

where the $Sh(x;y)$'s denote shuffles of the sequences x and y. The first and last shuffles here may be realized in a unique way as part of a shuffle of E_r as $Sh(\alpha; L(\beta_2))$ and $Sh(\gamma; R(\beta_1))$ respectively. Thus we must prove that the middle shuffle

$$Sh(L(\beta'), M(\beta'), R(\beta'); M(\beta''))$$

is uniquely realizable as a shuffle

$$Sh(L(\beta_1), M(\beta_1); M(\beta_2), R(\beta_2)),$$

of the remaining parts of β_1 and β_2. Note that this is precisely of the same form as our original problem. Thus we will be done by induction on n provided the length has been reduced by the first step. The length is not reduced if and only if both α and β are empty, so that our original series were $E_r = (\beta_1;\beta_2)$ and $E_s = (\beta';\beta'')$. But the shuffles of these two are in obvious bijection. Note that if $n=1$ in the original problem, then β and one of α and γ is empty, so that there is nothing to prove.

To prove the remaining assertions of the Proposition, we will show that, given a sequence which arises from a shuffle of E_r either we can uniquely recover the shuffle from the given sequence, and hence the sequence will have multiplicity one as an exponent, or the sequence arises as a shuffle of E_s. Moreover, in the second case, we will show that every shuffle of E_r which yields the given sequence is one of the shuffles which is matched to a shuffle of E_s by the algorithm above. Note that the second case occurs, i.e., the given sequence may be obtained by a shuffle of E_s, if and only if it contains the subsequence (α,β,γ) and the complement of this subsequence is β. These assertions will again be proved by induction on n.

Consider a sequence X which arises by an arbitrary shuffle of E_r. Let $L(X)$ be the set of entries of X from β which occur in X to the left of some element of α and let $R(X)$ be the set of entries of X from β which occur to the right of some element of γ. Let $L(\beta)$ and $R(\beta)$ denote the corresponding subsets of β. Note that $L(\beta)$ (resp. $R(\beta)$) it an initial (resp. final) subsequence of β, so that we may write

$$\beta = L(\beta)\beta^L = \beta^R R(\beta)$$

for certain subsequences β^L and β^R. Note that elements of $L(X)$ must have come from β_2 while elements of $R(X)$ must have come from β_1.

Now if some element of γ occurs to the left of some element of α, then $L(\beta) = \beta = R(\beta)$. In this case there is a unique shuffle which yields X and X cannot arise from a shuffle of E_s.

Similarly, if $L(\beta)\cup R(\beta) = \beta$, then the source of all remaining entries of X which come from β is determined uniquely. Thus, again, there is a unique shuffle which gives rise to X.

We may then suppose that all elements of α in X lie to the left of all elements of γ, and that

$$\beta = (L(\beta), M(\beta), R(\beta)),$$

for some non-empty sequence $M(\beta)$. The sequence X then has the form

$$(Sh(\alpha; L(\beta_2)), Sh(L(\beta_1), M(\beta_1); M(\beta_2), R(\beta_2)), Sh(\gamma; R(\beta_1))),$$

in the notation introduced above.

Reversing the procedure used before, we can realize the first and last of these shuffles in a unique way as part of a shuffle of E_s, i.e., as $Sh(\alpha; L(\beta''))$ and $Sh(\gamma; R(\beta''))$. It then remains to prove that the remaining sequence

$$Sh(L(\beta_1), M(\beta_1); M(\beta_2), R(\beta_2)),$$

either has a unique expression as a shuffle of

$$(L(\beta_1), M(\beta_1); M(\beta_2), R(\beta_2)),$$

or that every shuffle which yields this sequence is matched to a unique shuffle of

$$(L(\beta'), M(\beta'), R(\beta'); M(\beta'')).$$

Once again, this problem has the same form as our original one so that we are done, by induction on n, provided that we have indeed shortened the sequence X by the steps above.

To complete the proof, we must observe what happens when n is not reduced in the first step, i.e., when α and γ are empty. In this case, $E_r = (\beta_1; \beta_2)$ and $E_s = (\beta; \beta)$, so that the two sets of shuffles coincide, and we are done. ∎

Proposition 6.2 has several useful consequences.

First let us assume that we are in the case $\chi \neq 1$, $\chi^2 = 1$ with $s_0 = \frac{m}{2} - \rho_n$ for $n+1 < m \leq 2n$. Let V_1, V_2, $V_{1,0}$ and $V_{2,0}$ be as above, and let R be the set of exponents of $I_n(s_0, \chi)$ which arise from shuffles of $E_r(m)$'s with $0 \leq r \leq \frac{m}{2} - 1$. Let S be the set of exponents which arise from shuffles of $E_r(m)$'s with $\frac{m}{2} \leq r \leq n$. Elements of both of these sets are counted with multiplicity. Finally, let D denote the set of exponents $R - S$, which is well defined by (i) of Proposition 6.2 and which is multiplicity free by (ii) of that Proposition.

PROPOSITION 6.4:

(i) *R is the set of exponents of $R_n(V_i)$, for $i = 1, 2$.*

(ii) *S is the set of exponents of $R_n(V_{i,0})$, for $i = 1, 2$.*

(iii) *D is the set of exponents of $R_n(V_1) \cap R_n(V_2)$ and of $I_n(-s_0, \chi)/(R_n(V_{1,0}) \oplus R_n(V_{2,0}))$.*

Proof: Parts (i) and (ii) are just restatements of earlier results. To prove (iii) note that, since $\ker M_n^*(s_0) = R_n(V_1) \cap R_n(V_2)$, by Proposition 5.6, we have

$$R_n(V_1)/(R_n(V_1) \cap R_n(V_2)) \xrightarrow{\sim} R_n(V_{1,0}).$$

Our claim now follows from (i), (ii), Proposition 6.2 and the exactness of the Jacquet functor. ∎

PROPOSITION 6.5: *For m, s_0, etc., as above,*

$$\ker(M_n^*(-s_0)) = R_n(V_{1,0}) \oplus R_n(V_{2,0}),$$

and

$$\mathrm{Im}(M_n^*(-s_0)) = R_n(V_1) \cap R_n(V_2).$$

Proof: Let U be the unipotent radical of our fixed Borel subgroup B. The U-Jacquet module $I_n(s_0, \chi)_U$, which is a complex vector space of dimension 2^n, may then be decomposed according to the exponents, which give the action of the maximal split torus A. We then obtain a decomposition

$$I_n(s_0, \chi)_U = X \oplus Y,$$

stable under the action of A, where X is the subspace with exponents in S and Y is the subspace with exponents in D. Note that the simplicity of the exponents in D implies that there is a basis for Y, unique up to scalars, consisting of eigenvectors for the action of A . Since the exponents of $I_n(s, \chi)$ are holomorphic functions of s, there is an open neighborhood of s_0 on which the exponents interpolating those in D remain simple and disjoint from those interpolating exponents in $S = R - D$. Thus, for s in this neighborhood, we still have a decomposition

$$I_n(s, \chi)_U = X(s) \oplus Y(s).$$

Note that

$$Y = Y(s_0) = (R_n(V_1) \cap R_n(V_2))_U .$$

Similarly, we have a decomposition

$$I_n(-s_0, \chi)_U = X(-s_0) \oplus Y(-s_0)$$

and an extension of this

$$I_n(-s, \chi)_U = X(-s) \oplus Y(-s)$$

to a neighborhood of $-s_0$. Now the normalized intertwining operators induce operators $M_n^*(s, \chi)_U$ and $M_n^*(-s, \chi)_U$ on the U-Jacquet functors. If $\lambda = \lambda(s)$

is an exponent in $D(s)$ (the space of exponents extending those in D), and if $v(s) \in Y(s)$ (resp. $v(-s) \in Y(-s)$) is a corresponding eigenvector, chosen to depend holomorphically on s, we must have

$$M_n^*(s,\chi)_U v(s) = \mu(s) v(-s)$$

and

$$M_n^*(-s,\chi)_U v(-s) = \nu(s) v(s)$$

for some holomorphic functions $\mu(s)$ and $\nu(s)$. Note that $\mu(s)$ has a zero at $s = s_0$ since $R_n(V_1) \cap R_n(V_2)$ is the kernel of $M_n^*(s_0,\chi)$. On the other hand, the fact that

$$M_n^*(-s,\chi) \circ M_n^*(s,\chi) = \eta_n(s,\chi) \cdot \mathrm{Id},$$

implies that

$$\nu(s) \cdot \mu(s) = \eta(s)$$

has a *simple zero* at $s = s_0$ and hence that

$$\nu(s_0) \neq 0.$$

Thus $M_n^*(-s_0,\chi)_U$ is non-zero on $Y(-s_0)$. Since we already know that $R_n(V_{1,0}) \oplus R_n(V_{2,0})$ lies in the kernel of $M_n^*(-s_0,\chi)$, we must have

$$X(-s_0) = (R_n(V_{1,0}) \oplus R_n(V_{2,0}))_U = \ker(M_n^*(-s_0,\chi)_U),$$

and hence that

$$R_n(V_{1,0}) \oplus R_n(V_{2,0}) = \ker(M_n^*(-s_0,\chi))$$

by the exactness of the U-Jacquet functor.

It follows immediately that

$$\mathrm{Im}(M_n^*(-s_0,\chi)_U) = (R_n(V_1) \cap R_n(V_2))_U$$

and hence that

$$R_n(V_1) \cap R_n(V_2) = \mathrm{Im}(M_n^*(-s_0,\chi)),$$

as claimed. ∎

Finally we can finish our determination of the composition series of $I_n(s_0,\chi)$.

PROPOSITION 6.6: *For $\chi \neq 1$ and for $n+1 < m \leq 2n$,*

$$R_n(V_1) \cap R_n(V_2)$$

is irreducible.

Proof: Suppose that $W \subset R_n(V_1) \cap R_n(V_2)$ is a non-zero irreducible submodule. Then the argument of the proof of Theorem 2.6 implies that there is a non-zero intertwining operator

$$T : I_n(-s_0, \chi) \longrightarrow W \subset I_n(s_0, \chi).$$

By (i) of Proposition 2.2, T must be a non-zero multiple of $M_n^*(-s_0, \chi)$. But we have just seen that the image of $M_n^*(-s_0, \chi)$ is $R_n(V_1) \cap R_n(V_2)$. ∎

We next turn to the case $\chi = 1$ and assume that $n+1 < m < 2n$. As before, let V_1 be the split form and let V_2 be the quaternionic form of dimension m. Also let $V_{1,0}$ and $V_{2,0}$ be the complementary forms of dimension $m' = 2n+2-m$. Let R and S be the set of exponents defined above (note that the exponents of $I_n(s,\chi)$ do not depend on χ), and let R_0 be the subset of R consisting of those exponents which arise as shuffles of E_r's with $0 \leq r \leq \frac{m}{2} - 2$. Also let $S_0 \subset S$ be the subset consisting of those exponents which arise as shuffles of E_r's with $r > \frac{m}{2}$. By Lemma 6.3, note that exponents in S_0 can only match exponents in R_0. Applying Proposition 6.2 and the argument of the proof of Lemma 6.1, we have

PROPOSITION 6.7:

(i) *$R \cup (S - S_0)$ is the set of exponents of $R_n(V_1)$ and R_0 is the set of exponents of $R_n(V_2)$.*

(ii) *$S \cup (R - R_0)$ is the set of exponents of $R_n(V_{1,0})$ and S_0 is the set of exponents of $R_n(V_{2,0})$.*

(iii) *$R_0 \supset S_0$ and $R_0 - S_0$ is simple and disjoint from S_0.*

(iv) *$R - R_0 \supset S - S_0$ and $(R - R_0) - (S - S_0)$ is simple and disjoint from $S - S_0$.*

(v) *$R_0 - S_0$ is the set of exponents of $R_n(V_1) \cap R_n(V_2)$.*

By the same arguments as before Proposition 6.7 yields:

PROPOSITION 6.8: *For $\chi = 1$ and for $n+1 < m < 2n$,*

$$\ker(M_n^*(-s_0)) = R_n(V_{1,0}) \oplus R_n(V_{2,0}),$$

and

$$\mathrm{Im}(M_n^*(-s_0)) = R_n(V_1) \cap R_n(V_2).$$

PROPOSITION 6.9: *For $\chi = 1$ and for $n+1 < m < 2n$,*

$$R_n(V_1) \cap R_n(V_2)$$

is irreducible.

Finally, we consider the cases $\chi = 1$ and $m = 2n$ of $2n+2$.

First suppose that $m = 2n$. Then, since the $E_r(m)$'s have the form

$$(1-n, 2-n, \ldots, r-n; 0, \ldots, n-r-1),$$

distinct shuffles of a given E_r yield distinct exponents, and the only possible overlaps occur for $r + r' = 2n-1$, i.e., for $r = n-1$ and $r' = n$. In fact, the only non-simple exponent is

$$(1-n, 2-n, \ldots, 0),$$

which occurs with multiplicity 2. The shuffles of E_r for $r = n$ and $n-1$ are the exponents of $R_n(V_{1,0})$ while the exponents of $R_n(V_2)$ are all simple, and do not overlap with those of $R_n(V_{1,0})$.

Next suppose that $m = 2n+2$. Then all of the exponents are simple, and the only exponent which does not occur in $R_n(V_2)$ is

$$(-n, 1-n, \ldots, -1) = -\rho_B.$$

This is the unique exponent of the trivial subrepresentation $R_n(V_{1,0}) = \mathbb{C}$ of $I_n(-\rho_n, 1)$.

With this information we can complete the proof of Propositions 5.5 and 5.6.

End of the proof of Proposition 5.5 and Proposition 5.6: Recall that V_1 is the split space and let V_2 is the quaternionic space of dimension $m = 2n$ or $2n+2$. Then there is no complementary space for V_2. By (ii) of the Proposition 4.4 of [10], we have

$$\dim \mathrm{Hom}_G(R_n(V_2), I_n(-s_0, \chi)) = 0,$$

while by (iii) of Proposition 3.4 above, we have $R_n(V_1) = I_n(s_0, 1)$. Since

$$b_n(s,1)^{-1} b_n(-s,1)^{-1}$$

has a simple zero at s_0, the K type argument of the first part of the proof of Proposition 5.5 again shows that $R_n(V_{1,0})$ is contained in the image of $M_n^*(s_0)$. But now the faithfulness of the U-Jacquet functor on constituents and description of the exponents just given imply that $R_n(V_{1,0})$ must be precisely the image of $M_n^*(s_0)$ and $R_n(V_2)$ must be precisely its kernel. ∎

Finally, we have the analogue of Propositions 6.6 and 6.9.

PROPOSITION 6.10: *If $m = 2n$ or $2n+2$ and $\chi = 1$ then the submodule $R_n(V_2)$ associated to the quaternionic form V_2 of dimension m is irreducible.*

The proof is the same.

The following is a very special case of the Howe duality conjecture [**6**,**21**]. Note that we allow residue characteristic 2.

COROLLARY 6.11: *$R_n(V)$ has a unique irreducible quotient.*

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